APMA2560: Numerical Solutions to PDE II

Alexey Izmailov

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Introduction

These are some of my notes from Professor Mark Ainsworth's course, as taught in Spring '22, at Brown University. A few lectures are missing due to illnesses and my neglect of transcribing algorithms to LaTeX, but the theoretical aspects of classical finite element theory, with Professor Ainsworth's perspective, are here.

Although the official textbook for the course was *Numerical Solution of Partial Differential Equations by the Finite Element Method* by Claes Johnson, neither did this course proceed through the book linearly nor were all of the topics taught in this course to be found in that textbook.

Lecture 1 [1/26]

The main premise of this lecture is to convince us that the traditional approach to PDEs is wrong and in more than just the way that the considered equations are not representative of real PDE work. Therefore, the main objective of this lecture is to introduce the variational calculus and show how a variational formulation of a PDE allows us to obtain solutions that have meaningful physical interpretations which may otherwise be inaccessible in the "classical" formulation of PDE problems because of the strict requirements placed on the domain of interest as well as on the solution function.

To begin, we consider the problem of a deflection of a membrane. Given a two dimensional domain Ω with domain boundary indicated by $\partial\Omega$, suppose that the force applied at a point $(x, y) \in \Omega$ is given by $f(x, y)$. We wish to study the deflection of the membrane, given by $u(x, y)$. The relevant physical principle is that the membrane adopts a deflection of minimal energy.

Consider a quadrilateral subset of Ω given by $ABCD$ that maps to $A'B'C'D'$ under the deflection. We make the assumption that $u = 0$ for all points on $\partial\Omega$. Let

$$A : (x, y),\ B : (x + \delta x, y),\ C : (x, y + \delta y),\ D : (x + \delta x, y + \delta y)$$

and

$$A' : (x, y, u(x, y)),\ B' : (x + \delta x, y, u(x + \delta x, y)),$$
$$C' : (x, y + \delta y, u(x, y + \delta y)),\ D' : (x + \delta x, y + \delta y, u(x + \delta x, y + \delta y)).$$

Assuming $\delta \to 0$, we have that

$$|A'B'C'D'| \approx |\overrightarrow{A'C'} \times \overrightarrow{A'B'}|$$

where

$$\overrightarrow{A'C'} = (\delta x, 0, u(x + \delta x, y) - u(x, y)),$$
$$\overrightarrow{A'B'} = (0, \delta y, u(x, y + \delta y) - u(x, y)).$$

Then

$$\overrightarrow{A'C'} \times \overrightarrow{A'B'} = \begin{bmatrix} -\delta y(u(x + \delta x, y) - u(x, y)) \\ -\delta x(u(x, y + \delta y) - u(x, y)) \\ \delta x \delta y \end{bmatrix}$$

from which

$$|\overrightarrow{A'C'} \times \overrightarrow{A'B'}| = \delta x \delta y \left\{ 1 + \left(\frac{u(x+\delta x, y) - u(x,y)}{\delta x} \right)^2 + \left(\frac{u(x, y+\delta y) - u(x,y)}{\delta y} \right) \right\}^{1/2}.$$

Thus, the change in area of the quadrilateral under the deflection is

$$\delta x \delta y \left[\left\{ 1 + \left(\frac{u(x+\delta x, y) - u(x,y)}{\delta x} \right)^2 + \left(\frac{u(x, y+\delta y) - u(x,y)}{\delta y} \right) \right\}^{1/2} - 1 \right].$$

Summing across all such quadrilaterals in Ω and then taking $\delta x, \delta y \to 0$, we have that the energy stored in the membrane obtained from work done by the external force is given by

$$\int_\Omega dxdy \left[\sqrt{1 + |\nabla u|^2} - 1 \right].$$

Analogously, the total work done is given by

$$\int_\Omega f(x,y) u(x,y) dxdy$$

so the potential energy corresponding to the state, which is a functional, is given by

$$J(u) = \int_\Omega dxdy \left[\sqrt{1 + |\nabla u|^2} - 1 \right] - \int_\Omega f(x,y) u(x,y) dxdy.$$

To determine u, in accordance with the minimum energy principle, we are trying to determine the u that minimizes $J(v)$ for all admissible (for example, not tearing the membrane) displacements v. To make this specific, define the space of functionals

$$V = \{ v : \Omega \to \mathbb{R} | v = 0 \text{ on } \partial\Omega, J(v) < \infty \}.$$

Therefore, we have recast the problem: find $u \in V$ such that $J(u) \leq J(v)$ for all $v \in V$. Let $v \in V$ such that $v = u + \varepsilon w$ for $w \in V$ and $\varepsilon \in \mathbb{R}$. Intuitively, this is a (small) perturbation of u. The problem is therefore equivalent to finding u such that

$$J(u) \leq J(u + \varepsilon w) \; \forall w \in V, \varepsilon \in \mathbb{R}.$$

Performing a series expansion,

$$J(u + \varepsilon w) = \int_\Omega dxdy \left[\sqrt{1 + |\nabla u + \varepsilon \nabla w|^2} - 1 \right] - \int_\Omega f(x,y)(u + \varepsilon w) dxdy.$$

Now assuming $|\nabla u| \ll 1$, meaning we are working with small deflections,

$$\sqrt{1 + |\nabla u|^2} \approx 1 + \frac{1}{2} |\nabla u|^2 + \cdots$$

Therefore,

$$J(u + \varepsilon w) \approx \frac{1}{2} \int_\Omega |\nabla u|^2 + \varepsilon \int_\Omega \nabla u \cdot \nabla w + \frac{1}{2} \varepsilon^2 \int_\Omega |\nabla w|^2 - \int_\Omega f(u) - \varepsilon \int_\Omega f w$$

where the variables that are being integrated across are omitted for sake of brevity. Thus, using the definition of $J(u)$,

$$J(u + \varepsilon w) \approx J(u) + \varepsilon \left\{ \int_\Omega \nabla u \cdot \nabla w - \int_\Omega f w \right\} + \mathcal{O}(\varepsilon^2).$$

Thus the condition of minimality $J(u) \leq J(u + \varepsilon w)$ leads to

$$0 \leq \varepsilon \left\{ \int_\Omega \nabla u \cdot \nabla w - \int_\Omega f w \right\} + \mathcal{O}(\varepsilon^2).$$

Now considering individually the cases that $\varepsilon > 0$ and $\varepsilon < 0$, we obtain the variational problem of finding $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla w = \int_\Omega f w$$

for all $w \in V$. This has the physical interpretation of "the principle of virtual work". Now by integration by parts, in the spirit of Nirenberg's "integration by parts",

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla w &= \int_\Omega \nabla \cdot (w \nabla u) - \int_\Omega w \Delta u \\ &= \int_{\partial\Omega} \hat{n} \cdot w \nabla u - \int_\Omega w \Delta u \\ &= -\int_\Omega w \Delta u\end{aligned}$$

where we have used the divergence theorem and the fact that $u = 0$ on $\partial\Omega$. Provided u is sufficiently smooth, for all $w \in V$,

$$\int_\Omega w(f + \Delta u) = 0.$$

Now suppose $f + \Delta u$ is sufficiently smooth. Then if $f + \Delta u = 0$, we obtain the classical Poisson PDE. Alternatively, if $f + \Delta u \neq 0$, then by the smoothness assumption, there is a ball around the point at which $f + \Delta u$ attains a non-zero value. Since equality must hold for all $w \in V$, we can choose a w such that it is 0 everywhere except for the support of $f + \Delta u$, which leads to a contradiction on the equality. Therefore, it must be the case that $f + \Delta u = 0$. This is alternatively known as the fundamental lemma of variational calculus, which generalizes appreciably.

Now the classical formulation of the Poisson PDE requires that $u \in \mathcal{C}^2(\Omega)$ because of the necessity for the existence of Δu. However, note that in the variational formulation

$$\int_\Omega \nabla u \cdot \nabla w = \int_\Omega f w,$$

the only restriction is that $u \in \mathcal{C}^1(\Omega)$. The conclusion is that the classical formulation narrows the set of physically meaningful solutions that are being solved for.

As a palpable example, consider a one-dimensional string under tension strung between two poles which are L distance away from the origin in opposite directions. Suppose that a weight is attached at the midpoint. The equation for this is the one-dimensional Poisson equation:

$$-u'' = f.$$

Suppose the actual solution is given by

$$u(x, y) = \alpha \begin{cases} -x - L & -L < x < 0 \\ x - L & 0 < x < L \end{cases}$$

But the classical formulation via the Poisson equation does not admit a solution because of the point of discontinuity at the origin while this solution does satisfy the variational problem. In fact,

$$\int_{-L}^{L} u'w' = \int_{-L}^{L} f w = -W w(0)$$

for all $w \in V$. This can be solved for $\alpha = W/2$. Once again, the conclusion is that some problems do not admit a classical point-wise solution but have a variational solution, showing the advantage conferred by a variational statement of the PDE.

Lecture 2 [2/2]

This lecture started with considering the bending of a doubly-fixed 3D elastic beam. That means that there are built-in supports, so the beam cannot move at $x = 0, \ell$. Let $f(x)$ be the force per unit length applied at $x \in (0, \ell)$, $u(x)$ be the deflection of the beam from the undisplaced configuration. In the following, we assume that there are only small deflections, meaning $\mathcal{O}(u^2)$ terms are negligible. Without this assumption, we are in the case of Euler-Bernoulli beams. The mid-surface of the beam, alternatively known as the neutral axis, goes through the middle of the beam. Tension changes to compression when crossing the neutral axis. Thus, we can be more specific and say $u(x)$ specifies the deflection of the mid-surface.

Consider a portion of the beam between x and $x + \delta x$ around a bend in the deformed beam. The relevant arclength is δs over an angle $\delta\theta$ and can be thought of as the length of the neutral axis in the section of material. Let $\rho(x)$ be the radius of curvature at x. If the beam is totally flat at its initial configuration, $\rho = \infty$. Clearly $\rho(x)$ has a dependence on $u(x)$. Under the Kirchhoff-Love Hypothesis, the individual fibers of the material are all perpendicular to the neutral axis. Now consider a section of material a distance y above the neutral axis with thickness δy. If we are interested in computing the energy stored in a fiber located y above the neutral axis located in the region described by

$$(x, x + \delta x) \times (y, y + \delta y) \times (z, z + \delta z).$$

observe that the energy arises from either extension or compression of every fiber. Quantitatively, since hte length of the deflected fiber is determined by $(\rho(x) + y)\delta\theta$, the strain, which is the ratio of increase in length to the original length of the material, is given by

$$\text{strain} = \frac{(\rho(x) + y)\delta\theta - \rho(x)\delta\theta)}{\rho(x)\delta\theta} = \frac{y}{\rho(x)}.$$

Hooke's law gives that stress, which is force per unit area, in the fiber is directly proportional to strain. This implies that

$$\text{stress} = \frac{Ey}{\rho}$$

where E is the Young's modulus of the material. Thus,

$$\text{force in fiber} = \frac{Ey}{\rho}\delta y \delta z.$$

Combining this,

$$\text{energy stored in material} = \frac{1}{2}\left(\frac{Ey}{\rho}\delta y \delta z\right) y\delta\theta.$$

Let

$$\mathcal{C} = \{(y, z) : \text{in beam}\}.$$

Then taking $\delta y, \delta z \to 0$,

$$\sum_{(y,z)\in\mathcal{C}} \frac{1}{2}\frac{E}{\rho} y^2 \delta y \delta z \delta\theta = \frac{E}{2}\int_{\mathcal{C}} y^2 dydz \frac{\delta\theta}{\rho(x)} = \frac{1}{2} EI \frac{\delta\theta}{\rho(x)}$$

where

$$I = \int_{\mathcal{C}} y^2 dydz$$

is the moment of inertia for the beam. This of course depends on the beam geometry. We also call EI the "flexural rigidity". We conclude that

$$\text{total energy of bending is} = \sum_{0<x<\ell} \frac{1}{2} EI \frac{\delta\theta}{\rho(x)}.$$

Now writing in terms of x: $\theta(x) \approx \tan\theta(x) = u'(x)$. Then by the definition of curvature,

$$\frac{1}{\rho(x)} = \frac{\delta\theta}{\delta s} = \frac{d\theta}{ds} \approx \frac{d\theta}{dx} = u''(x).$$

This implies

$$\frac{1}{2}EI\frac{\delta\theta}{\rho(x)} = \frac{1}{2}EIu''(x)^2\delta x$$

because $\delta\theta \approx u''(x)\delta x$. Because

$$\text{energy stored in beam} = \int_0^\ell \frac{1}{2}EIu''(x)dx,$$

and the external force $f(x)$ per unit length does work where there is deflection, the work is

$$\int_0^\ell f(x)u(x)dx,$$

we can define the potential energy functional

$$J(v) = \frac{1}{2}EI\int_0^\ell u''(x)^2dx - \int_0^\ell f(x)u(x)dx.$$

The negative sign comes in because energy is conserved – the force does work in the direction of motion determined by the gravitational force. An alternative way to see this is that

$$\text{const} = J(u) + \int_0^\ell f(x)u(x)dx.$$

Now this functional $J : V \to \mathbb{R}$ where V is the space of admissible deflections:

$$\begin{aligned} V &= \{v : [0,\ell] \to \mathbb{R}, v(0) = v'(0) = 0, v(\ell) = v'(\ell) = 0, J(v) < \infty\} \\ &= \}v : [0,\ell] \to \mathbb{R}, v(0) = v'(0) = v(\ell) = v'(\ell) = 0, J(v) < \infty\} \\ &= H_0^2(0,\ell) \end{aligned}$$

where H_0^2 is the Sobolev space of index 2 with homogenous boundary conditions being 0. Now the Sobolev space $H_0^0 = L^2(0,\ell)$. So we have the problem

$$u \in V : J(u) \leq J(v)\ \forall v \in V.$$

To pass to the equation, we can do the usual perturbation method, as in Lecture 1, or to just use the Euler condition for the extrema:

$$\frac{d}{d\varepsilon}J(u+\varepsilon w)\bigg|_{\varepsilon=0} \implies EI\int_0^\ell u''(x)v''(x)dx = \int_0^\ell fv(x)dx.$$

This is just the principle of least action. Integrating the expression above by parts,

$$\int_0^\ell fv = [EI(u''v' - u'''v)]_0^\ell + \int_0^\ell EIu^{(4)}vdx$$

where we can get rid of the first term because of the boundary conditions. Thus for all $v \in H_0^2(0,\ell)$ and assuming proper smoothness,

$$\int_0^\ell (f - EIu^{(4)})v = 0$$

for every $v \in V$, which gives that

$$EIu^{(4)}(x) = f(x),\ x \in (0,\ell)$$

by the Fundamental Lemma. This is known as the beam equation. Note that this requires the assumption that $u \in C^4(0,1), f \in C(0,1)$.

Now suppose we ignore some of the boundary conditions. For instance, by letting the right hand side of the beam situation not fixed. Thus, define

$$V = \{v : (0,\ell) \to \mathbb{R}, v'', v', v \in L^2(0,\ell), v(0) = v'(0) = 0\}.$$

This removes the ease with which we got rid of the term in the integration by parts above, but the variational problem still holds true. We can recover the terms through a multi-step process and obtain new boundary conditions in the process.

- Step 1. Assume $v(\ell) = v'(\ell) = 0$, which reduces to the same problem. Note that this set of functionals is a strict subset of the V defined above.
- Step 2. Back substitute into

$$\int_0^\ell fv = \int_0^\ell EIu^{(4)}v + EI(u'''(\ell)v(\ell) - u''(\ell)v'(\ell)).$$

We have that

$$EI(u'''(\ell)v(\ell) - u''(\ell)v'(\ell)) = 0$$

for all $v \in V$.

  - Step 2a. Choose $v \in V$ such that $v(\ell) = 0$. Then

$$EIu''(\ell) = 0$$

  -
  - Step 2b. Choose $v'(\ell) = 0$ so

$$EIu'''(\ell)v(\ell) = 0 \implies EIu'''(\ell) = 0.$$

So this process gives us natural boundary conditions $u''(\ell) = 0, u'''(\ell) = 0$ while the essential boundary conditions were established in the formulation of the problem with the constraint $u(0) = u'(0) = 0$. This process can be extended to when the boundary condition on the left-hand side is relaxed to just $u(0) = 0$ and nothing else. The variational problem is still true and it is possible to derive a new set of boundary conditions. The fact that a new finite difference scheme must be developed and proved for every set of boundary conditions indicates that that approach is not too practical.

Lecture 3 [2/9]

First, a rehash of the steps at the end for obtaining natural boundary conditions. Let the variational equation work on

$$V = \{v : (0, \ell) \to \mathbb{R}, v, v', v'' \in L^2(0, \ell), v(0) = v'(0) = 0\}$$

which has the physical meaning of the beam from last lecture having one end detached from a wall. The differential equation is

$$\int_0^\ell fv = EI \int_0^\ell u^{(IV)}v + EI[u''(\ell)v'(\ell) - u'''(\ell)v(\ell)].$$

Consider three cases.

- (1) $v(\ell) = v'(\ell) = 0$. Then the fundamental theorem gives

$$f = EIu^{(IV)} \text{ on } (0, \ell).$$

- (2) The variational problem is still true when $v(\ell) = 0, v'(\ell) = 0$, which gives

$$0 = EIu''(\ell)v'(\ell)$$

- (3) Analogously, the case $v(\ell) \neq 0, v'(\ell) = 0$ yields

$$0 = EIu'''(\ell)v(\ell).$$

We conclude that the natural boundary conditions are given by $u''(\ell)v'(\ell) = u'''(\ell)v(\ell) = 0$.

Now suppose the beam is still detached on one end and a couple force M is applied at $x = \ell$. The new energy functional takes into account the work done by M and is given by

$$J(v) = \frac{1}{2}EI\int_0^\ell v''(x)^2 - \int_0^\ell fv - M\theta(\ell).$$

Using the approximation $\theta(\ell) \approx \tan\theta(\ell) = u'(\ell)$,

$$J(u) = \frac{1}{2}EI\int_0^\ell u''(x)^2 - \int_0^| ellfu - Mu'(\ell).$$

To attain the variational equation,

$$\begin{aligned} \frac{d}{d\varepsilon}J(u+\varepsilon v)\bigg|_{\varepsilon=0} &= 0 \\ EI\int_0^\ell u''(\ell)v''(\ell) &= \int_0^\ell fv + Mv'(\ell)\ \forall v \in V. \end{aligned}$$

This gives an IBP again:

$$\int_0^\ell fv = \int_0^\ell EIu^{(IV)} + EI(u''(\ell)v'(\ell) - u'''(\ell)v(\ell)) + Mv'(\ell).$$

Using the same steps described earlier, we obtain the natural boundary conditions

$$u''(\ell) = -\frac{M}{EI},\ u'''(\ell) = 0.$$

Pointing a force F at the free end of the beam (meaning an up-down force, not a couple force, which acts angularly),

$$J(u) = \frac{1}{2}EI\int_0^\ell u''(x)^2 - \int_0^\ell fu - Fu(\ell).$$

By similar analysis, $u''(\ell) = 0, u'''(\ell) = Q$.

The main takeaway is that the variational equation encodes boundary conditions. Different versions may appear to have two boundary conditions, which is only problematic for the well-posedness of the ODE, but it turns out that the variational equation actually has two extra "natural" boundary conditions in its definition.

We now introduce a Timoshenko beam model. Before, the Kirchkoff-Love hypothesis assumed that $\theta(x) = u'(x)$, meaning all fibers of the beam were normal to the neutral surface of the beam. However, the Timoshenko beam discards this assumption and says that the deformation at a point is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -y\theta(x) \\ u(x) \end{bmatrix}.$$

The question then becomes about determining $u(x)$ and $\theta(x)$. This model involves the physical concept of shearing. The strains are defined as

$$\begin{aligned} \varepsilon_{xx} &= -y\theta'(x) \\ \varepsilon_{yy} &= 0 \\ 2\varepsilon_{xy} &= u'(x) - \theta(x). \end{aligned}$$

In the Kirchoff-Love hypothesis, $u'(x) = \theta(x)$, so $\varepsilon_{xy} = 0$. The stresses are given by

$$\begin{aligned} \sigma_{xx} &= E\varepsilon_{xx} = -Ey\theta'(x) \\ \sigma_{yy} &= E\varepsilon_{yy} = 0 \\ \sigma_{xy} &= 2G\varepsilon_{xy} = G(u'(x) - \theta(x)) \end{aligned}$$

where E is the elastic modulus and G is the shear modulus. Then the internal energy in the beam at $(x, x+\delta x), (y, y+\delta y), (z, z+\delta z)$ is given by

$$\frac{1}{2}\delta x\delta y\delta z\{\sigma_{xx}\varepsilon_{xx} + 2\sigma_{xy}\varepsilon_{xx}\} =$$

$$\frac{1}{2}\delta x\delta y\delta z\{Ey^2(\theta(x))^2 + G(u'(x)-\theta(x))^2\} \implies$$

$$\int_{(y,z)\in\mathcal{C}} \frac{1}{2}\delta x\delta y\delta z\{Ey^2(\theta(x))^2 + G(u'(x)-\theta(x))^2\} = \frac{1}{2}\delta x\{EI(\theta'(x))^2 + GA(u'(x)-\theta(x))^2\}$$

where the integral is over a cross-section and

$$I = \int y^2 dydz,\ A = \int_{\mathcal{C}} dydz$$

are the moment of inertia and area of the cross section, respectively. Thus the potential energy is given by

$$J(u,\theta) = \frac{1}{2}EI\int_0^\ell ((\theta'(x))^2 + \frac{1}{2}GA\int_0^\ell (u'-\theta)^2 - \int_0^\ell fu.$$

Observe that assuming the Kirchoff-Love hypothesis, the above is the same energy functional as before. Moreover,

$$\frac{I}{A} = \frac{\int y^2 dydz}{\int dydz} \in \mathcal{O}(t^2)$$

where t is the thickness of the beam. It follows that if the beam is very thin, we are interested in minimizing the A term more than the I term. This minimization problem can be solved approximately with $u' = \theta$, which reduces to the Euler-Bernoulli beam from before. That is to say that if the beam is thin, Timoshenko isn't necessary. The model is best applied in a setting with a thicker beam.

The admissible set is

$$\{(\psi, v) : \psi, \psi' \in L^2, v, v' \in L^2, u(0) = u(\ell) = 0, \theta(0) = \theta(\ell) = 0\}$$

which contains vector-valued functions and assumes that the end points are distorted. The variational problem is exactly

$$\frac{d}{d\varepsilon}J(u+\varepsilon v, \theta+\varepsilon\psi)\bigg|_{\varepsilon=0} \quad \forall(\psi, v) \in V$$

which is equivalent to

$$EI\int_0^\ell \theta'(x)\psi'(x) + GA\int_0^\ell (u'-\theta)(u'-\psi) = \int_0^\ell fv\ \forall(\psi, v) \in V.$$

If $v = 0, \psi \neq 0$,

$$EI\int_0^\ell \theta'\psi' - GA\int_0^\ell (u'-\theta)\psi = 0.$$

If $\theta \in C^2, u \in C^1$ and $\psi(0) = \psi(\ell) = 0$,

$$EI\int \theta''\psi - GA\int_0^\ell (u'-\theta)\psi = 0 \implies EI\theta'' + GA(u'-\theta) = 0.$$

If $v \neq 0, \psi = 0$,

$$GA\int_0^\ell (u'-\theta)v' = \int_0^\ell fv$$

then the IBP with $v(0) = v(\ell) = 0$ implies

$$-GA(u' - \theta)' = f.$$

So there are two differential equations but there are two functions v, ψ to play with.

Abstragating from the practical examples, a variational equation in general is given by

$$u \in V : a(u, v) = L(v) \; \forall v \in V$$

where V is the space of functions with smoothness depending on the problem and essential constraints. $L : V \to \mathbb{R}$ is a linear functional and is likely bounded by the physical setting. $a : V \times V \to \mathbb{R}$ is a bilinear form that is bounded in both arguments. Although a has been symmetric so far, the symmetry requirement is not necessarily assumed since apparently there are physical problems when the symmetry is broken.

To devise a numerical scheme to solve the variational problem, first make the restriction to the n-dimensional subspace V^n of the infinite dimensional space V. So the finite variational problem is given by

$$u_n \in V_n : a(u_n, v) = L(v), \; \forall v \in V_n.$$

This transition is known as a Galerkin scheme. Claims to be shown later: the scheme is guaranteed to be stable by inheritance from the original V problem and solving V_n solves V itself.

Lecture 4 [2/16]

Apparently the problem set from last week was a reconstruction of the Mindlin-Reissner plate theory, which is a general theory of the deformations and strains in a vibrating thick plate.

We now begin the construction of numerical schemes for the variational formulation of PDEs. We start with the variational equation given in general by

$$u \in V : a(u, v) = L(v) \quad \forall v \in V$$

where u is the solution, $a : V \times V \to \mathbb{R}$ is a bilinear form, v is called a test function, and $L(v)$ is a linear functional. Our claim is that for $\dim V < \infty$, the difficulty of solving the variational problem disappears.

Consider the finite variational problem for $V_n \subset V$ with $\dim V_n = n < \infty$

$$u_n \in V_n : a(u_n, v_n) = L(v_n) \quad \forall v_n \in V_n.$$

The finite dimensional problem is related to the general variational problem in the sense that if we compute $v_n \in V_n$, we can regard the Galerkin approximation $u_n \approx u$. To compute u_n, recall that since $\dim V_n$ is finite, there exists a basis $\{\varphi_j\}_{j=1}^n$ so that

$$V_n = \operatorname{span}\{\varphi_j\}_{j=1}^n \iff u_n = \sum_{j=1}^n \alpha_j \varphi_j$$

for a sequence of scalars $\{\alpha_j\}_{j=1}^n$. Thus, if the basis is known, the problem is reduced to computing exactly the $\vec{\alpha}$ vector. Substituting into the n-dimensional VE problem,

$$a\left(\sum_{j=1}^n \alpha_j \varphi_j, v_n\right) = L(v_n) \quad \forall v_n \in V_n.$$

and since all basis elements are part of V_n, this is equivalent to

$$a\left(\sum_{j=1}^n \alpha_j \varphi_j, \varphi_k\right) = L(\varphi_k) \quad \forall k \in \{1, 2, \ldots, n\}.$$

Thus we have that the n-dimensional problem is equivalent to solving the matrix problem

$$A\vec{\alpha} = \vec{L}$$

where $A_{ij} = a(\varphi_k, \varphi_j)$ and $L_k = L(\varphi_k)$. Note that this choice is because A is not necessarily symmetric – this occurs when $a(\cdot,\cdot)$ is not a symmetric bilinear form.

Without further assumptions, the matrix problem is not solvable because A is not invertible. To ensure that the problem is solvable, we make the follow assumptions: A is symmetric, which occurs if and only if $a(\cdot,\cdot)$ is a symmetric bilinear form, and that A is semi-positive definite. That is, $\vec{\beta}^T A\vec{\beta} \geq$ for all $\vec{\beta} \in \mathbb{R}^n$ and equality occurs if and only if $\vec{\beta} = \vec{0}$. The corresponding property that $a(\cdot,\cdot)$ is as follows:

$$\vec{\beta}^T A\vec{\beta} = \sum_{j,k} \beta_j a(\varphi_k, \varphi_j)\beta_k = a\left(\sum_k \beta_k\varphi_k, \sum_j \beta_j\varphi_j\right) = a(v_n, v_n)$$

where $v_n = \sum_k \beta_k\varphi_k$. So A is semi-positive definite if and only if $a(v_n, v_n) \geq 0$ for all $v_n \in V_n$ with equality occurring if and only if $v_n = 0$. The symmetry condition implies that $a(u_n, v_n) = a(v_n, u_n)$. This means, that in addition to being a bilinear form, $a(\cdot,\cdot)$ is semi-positive definite.

Recall that solvability of a square matrix occurs if and only if the solution is unique. Suppose $u_n, \bar{u}_n$ both satisfy the n-dimensional variational equation. Then

$$a(u_n - \bar{u}_n, v_n) = a(u_n, v_n) - a(\bar{u}_n, v_n) = L(v_n) - L(v_n) = 0 \quad \forall v_n \in V_n.$$

Choose $v_n = u_n - \bar{u}_n$, then

$$a(u_n - \bar{u}_n, u_n - \bar{u}_n) = 0 \iff u_n = \bar{u}_n$$

by the semi-positive definiteness condition.

Example 0.1. Consider $V = H^1(0,\ell)$ and

$$a(u, v) = \int_0^\ell u'(x)v'(x)dx.$$

Clear $a(\cdot,\cdot)$ is symmetric and

$$a(v, v) = \int_0^\ell (v'(x))^2 dx \geq 0$$

with equality occuring if and only if $v(x) =$ const. This is exactly the functional from the string example considered in class. If we specify that the string is fixed at $x = 0, \ell$ then $v(0) = v(\ell) = 0$ and the constant condition for equality in the semi-positive definite condition implies that $v(x) = 0$.

Example 0.2. In the Euler-Bernoulli beam,

$$a(u, v) = \int_0^\ell u''(x)v''(x)dx, \quad V = H^2(0,\ell)$$

with $a(\cdot,\cdot)$ being semi-positive definite and equality occuring if and only if $v'(x) = 0 \implies v(x) = mx + c$ for $m, c \in \mathbb{R}$. But the condition that $v(0) = v'(0) = 0$ implies that $m = c = 0 \implies v(x) = 0$.

Example 0.3. In the Timoshenko beam,

$$a(\{u,\varphi\}, \{v,\psi\}) = EI\int_0^e ll\varphi'(x)\psi'(x)dx + GA\int_0^\ell (u' - \varphi)(v' - \psi)dx$$

and

$$V = \{\{v,\psi\} : (0,\ell) \to \mathbb{R}^2, v, v', \psi, \psi' \in L^2, v(0) = \psi(0) = 0\}.$$

Again we see that $a(\cdot,\cdot)$ is symmetric with equality occurring if and only if $\psi'(x) = 0$ almost everywhere and $v'(x) = \psi(x) = 0$ almost everywhere. It follows that $\psi(x) =$ const $= 0$ so $v'(x) = 0$.

Example 0.4. In the 2D membrane problem,

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v$$

where $V = H_0^1(\Omega)$. Evidently

$$a(v, v) = ||\nabla v||^2 = 0 \iff \nabla v = 0 \implies v = \text{const}, \quad v = 0 \text{ on } \delta\Omega \implies v = 0.$$

An interesting historical remark, due to Professor Ainsworth's personal account: apparently the legendary engineer and one of the original pioneers of the finite element methods, Olgierd Zienkiewicz was opposed to Sobolev spaces, arguing that in his books, he can completely postulate all of these problems and prove results about them without mentioning the abstract object at all.

The above examples demonstrate that the hypothesis of the set-up of this problem is reasonable and the imposed conditions ensure that the Galerkin approximation u_n exists and is unique. Now suppose that the original variational equation has a unique solution. Then there exists a $u \in V$ such that $a(u,v) = L(v)$ for all $v \in V$. Since $V_n \subset V$, we can define a Galerkin scheme that solves

$$u_n \in V_n : a(u_n, v_n) = L(v_n) \quad \forall v_n \in V_n$$

where we can estimate the error $e = u - u_n \in V$ of the approximation using the energy norm

$$|||v||| = \sqrt{a(v,v)} \geq 0 \quad |||\alpha v||| = \sqrt{a(\alpha v, \alpha v)} = |\alpha|\sqrt{a(v,v)} = 0 \iff a(v,v) = 0 \iff v = 0.$$

This is "the one that makes physical sense". For additional motivation for this norm, consider bounding $|||e||| = |||u - u_n|||$. From the variational equation,

$$a(e,v) = a(u - u_n, v) = a(u,v) - a(u_n, v) = L(v) - a(u_n, v) \quad \forall v \in V.$$

Observe that if $v \in V_n \subset V$, then

$$a(e, v_n) = L(v_n) - L(v_n) = 0 \quad \forall v_n \in V_n$$

meaning $e \perp V_n$ – is orthogonal in to V_n in energy. This is known as the general property of Galerkin orthogonality, establishing that $|||e|||$ is the best possible approximation. More specifically,

$$|||e|||^2 = a(e,e) = a(e, u - v_n + v_n - u_n) = a(e, u - v_n) + a(e, v_n - u_n) = a(e, u - u_n) \leq |||e||| \cdot |||u - v_n||| \quad \forall v \in V_n.$$

If $e \neq 0$¡ then

$$|||e||| \leq |||u - v_n|||$$

for arbitrary $v_n \in V_n$. This is known as Cea's lemma, or the "best approximation property". u_n is the best approximation to u from the space V_n. This also ensures stability of the approximation.

We now begin the construction of our first Galerkin scheme. Consider the deflection of a string again with

$$a(u,v) = \int_0^\ell u'(x)v'(x)dx, \quad V = H_0^1(0\ell)$$

Suppose $L(v) = Pv(\bar{x}) + (f,v)$ where $P \in \mathbb{R}, f \in L_2, \bar{x} \in (0,\ell)$. We make a piece-wise linear approximation. Define the sequence of points

$$0 = x_0 < x_1 < x_2 < \cdots < x_{n+1} = \ell$$

that represent a mesh of "nodes" with $\dim V_n = n$. Let the height of v_n be given by α_i at the i-th node. Observe the degrees of freedom are given by values of the piecewise linear function at the nodes. Explicitly,

$$\alpha_j = v_n(x_j) \quad j = 1, 2, \ldots, n.$$

Once the degrees of freedom are specified, the basis is automatically determined:

$$v_n(x) = \sum_{j=1}^n \alpha_j \varphi_j(x)$$

For $x = x_k$,

$$v_n(x) = \sum_{j=1}^n \alpha_j \varphi_j(x_k) = \sum_{j=1}^n \alpha_j \delta_{jk}.$$

Intuitively, $\varphi_j(x)$ can be thought of as a "hat function". To compute the Galerkin approximation u_n, it is necessary to compute $A = \{a(\varphi_j, \varphi_k)\}, \vec{L}(\varphi_k)$. Observe that

$$a(\varphi_j, \varphi_k) = 0 \quad |j-k| \leq 1$$

because, if $h_j = x_j - x_{j-1}$, the only non-zero entries are given by

$$\begin{aligned} a(\varphi_j, \varphi_j) = \int_0^\ell (\varphi_j'(x))^2 dx &= \int_{x_{j-1}}^{x_j} (\varphi_j'(x))^2 dx + \int_{x_j}^{x_{j+1}} (\varphi_j'(x))^2 dx \\ &= \int_{x_{j-1}}^{x_j} \frac{1}{h_j^2} dx + \int_{x_j}^{x_{j+1}} \left(\frac{1}{h_{j+1}}\right)^2 dx \\ &= \frac{1}{h_j} + \frac{1}{h_{j+1}} \end{aligned}$$

as well as

$$a(\varphi_j, \varphi_{j+1}) = \int_{x_j}^{x_{j+1}} \left(-\frac{1}{h_{j+1}} \cdot \frac{1}{h_{j+1}}\right) dx = -\frac{1}{h_{j+1}}$$

and by symmetry,

$$a(\varphi_{j+1}, \varphi_j) = -\frac{1}{h_{j+1}}.$$

Thus we have the tri-diagonal matrix

$$A = \begin{bmatrix} \frac{1}{h_1} + \frac{1}{h_2} & -\frac{1}{h_2} & & & \\ -\frac{1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & -\frac{1}{h_3} & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & & & \end{bmatrix}$$

The entires of the load vector are given by $L(\varphi_k) = P\varphi_k(\bar{x}) + (f, \varphi_k) = P\delta_{k\ell} + \int_{x_{k-1}}^{x_k} f(x)\varphi_k(x)dx$ where we let $\bar{x} = x_\ell$ for some admissible ℓ for simplicity. The last integral term can be approximated with a midpoint rule quadrature as

$$\begin{aligned} L(\varphi_k) &= P\delta_{k\ell} + \int_{x_{k-1}}^{x_k} f(x)\varphi_k(x)dx + \int_{x_k}^{x_{k+1}} f(x)\varphi_k(x) \\ &\approx P\delta_{k\ell} + \frac{1}{2}\left[h_k f(x_{k-1/2}) + h_{k+1} f(x_{k+1/2})\right]. \end{aligned}$$

Then inverting A gives us $\{\alpha_j\}$.

We can make an a priori error estimate using Cea's Lemma. The error in the energy norm is given by

$$|||u - u_n||| \leq |||u - v_n||| \quad \forall v_n \in V_n.$$

Let $v = \Pi_n u \in V_n$ be a piecewise linear independent function. Local to a single interval, we want to approximate

$$\int_{x_{j-1}}^{x_j} ((u - \Pi_n u)'(x))^2 dx \quad \forall j.$$

Consider the interval $(0, h)$. We claim the following:

$$\mathcal{E}(x) = u(x) - p(x) = \frac{1}{h}\int_0^h K(x,t)u''(t)dt$$

where

$$K(x,t) = \begin{cases} t(x-h) & t < x \\ x(t-h) & x \geq t \end{cases}$$

is the Peano kernel. To see this, observe from integration by parts that

$$\int_0^\ell K(x,t)u''(t)dt = \int_0^x dtt(x-h)u''(t) + \int_x^h dtx(t-h)u''(t)$$
$$= h(u(x) - p(x)).$$

From this lemma,

$$\mathcal{E}(x)^2 \leq \frac{1}{h^2}\int_0^h K(x,t)^2 dt \cdot \int_0^h (u''(t))^2 dt$$
$$\leq \frac{1}{h^2}\int_0^h \int_0^h K(x,t)^2 dtdx ||u''||^2$$

from which it follows that

$$||\mathcal{E}(x)|| \leq \frac{h^2}{\sqrt{90}}||u''||.$$

Similarly, we can show that the term

$$\mathcal{E}'(x) = \frac{1}{h}\int_0^h \frac{\partial}{\partial x}K(x,t)u''(t)dt$$

satisfies

$$||\mathcal{E}'||^2 \leq \frac{1}{6}h^2||u''||^2.$$

The corresponding result over (x_{j-1}, x_j) is that

$$||(u - \Pi_n u)'||^2_{(x_{j-1},x_j)} \leq \frac{1}{6}h_j^2|u''||^2_{(x_{j-1},x_j)}$$

and summing over j, we obtain

$$||(u - \Pi_n u)'||^2 \leq \frac{1}{6}h_{\text{max}}^2||u''||^2_{(0,\ell)}.$$

This yields the following theorem: if $u \in H^2$, then

$$||e|| \leq \frac{1}{\sqrt{6}}h_{\text{max}}||u''||$$

where $h_{\text{max}} = \max_j h_j$. Thus, as $h_{\text{max}} \to 0$, we have $||e|| \to 0$, establishing convergence.

Lecture 5 [2/23]

The point of this lecture is to discuss the practical aspects of implementing an FEM code...

Lecture 6 [3/2]

The purpose of this lecture is to develop the theory of well-posedness for the variational problem. Consider the problem for finding

$$u \in V : a(u,v) = L(v) \quad \forall v \in V$$

with the associated minimization problem $J(u) \leq J(v)$ for all $v \in V$ where

$$J(v) = \frac{1}{2}a(v,v) - L(v).$$

Here, V is a vector space with norm $||v||_V$, $L : V \to \mathbb{R}$ is a linear functional and $a : V \times V \to \mathbb{R}$ is a bilinear symmetric form. Previously, we have shown that when $\dim V < \infty$ and $a(\cdot, \cdot) < \infty$, that the solution u can be computed. We are now interested in considering the case $\dim V = \infty$. Observe that in the finite dimensional case,

$$\frac{a(v,v)}{||v||^2} \geq \delta > 0,$$

but in the infinite dimensional case, $\delta \to 0$ so $a(v, v)$ (proofs?). In the finite dimensional case, we have shown that the corresponding matrix to $a(\cdot, \cdot)$ is semi-positive definite. Thus we need additional assumptions. We want there to be some $\Lambda > 0$ such that for all $v \in V$,

$$|L(v)| \leq \Lambda ||v||_V,$$

meaning $L(v)$ is a bounded linear functional. Additionally, assume

$$\frac{|a(u,v)|}{||u||_V ||v||_V} \leq M$$

which is called continuity and assume that there exists $\alpha > 0$ such that

$$\frac{a(v,v)}{||v||_V^2} \geq \alpha$$

which is called ellipticity/coercivity. We claim that under these conditions, the variational problem is well-posed (although in reality this isn't quite enough). As a side-note, the coercivity condition is actually stronger than being SPD.

We make the following argument to motivate the introduction of an additional assumption to those above. Suppose we wished to prove that the conditions above are enough to establish that there is a unique solution to the variational problem. Then from the definition of the corresponding minimization problem,

$$\begin{aligned}
J(v) = \frac{1}{2}a(v,v) - L(v) &\geq \frac{1}{2}\alpha ||v||_V^2 - L(v) \\
&\geq \frac{1}{2}\alpha ||v||_V^2 - \Lambda ||v||_V \\
&= \frac{1}{2}\alpha(||v||_V^2 - \frac{2\Lambda}{\alpha}||v||_V \\
&= \frac{1}{2}\alpha \left\{ \left(||v||_V^2 - \frac{\Lambda}{\alpha} \right)^2 - \frac{\Lambda^2}{\alpha^2} \right\} \\
&\geq \frac{1}{2}\alpha \left(-\frac{\Lambda^2}{\alpha^2} \right) = -\frac{\Lambda^2}{2\alpha} > -\infty,
\end{aligned}$$

which implies that

$$\Upsilon = \inf_{v \in V} J(v) \geq -\frac{\Lambda^2}{2\alpha}.$$

Let $n \in \mathbb{N}$, then by the definition of Υ, there exists a $u_n \in V$ such that

$$\begin{aligned}
\Upsilon \leq J(u_n) \leq \Upsilon + \frac{1}{n} \quad (*) \\
\Upsilon \leq J(u_m) \leq \Upsilon + \frac{1}{m}.
\end{aligned}$$

We claim that $||u_n - u_m|| \to 0$ for $m, n \to 0$, meaning this forms a Cauchy sequence. Observe that

$$\begin{aligned}
J(u_m) + J(u_n) - 2J\left(\frac{1}{2}u_m + \frac{1}{2}u_n\right) &= \frac{1}{2}\left\{\frac{1}{2}a(u_m, u_m) + \frac{1}{2}a(u_n, u_n) - a(u_n, u_m)\right\} \\
&= \frac{1}{4}a(u_m - u_n, u_m - u_n) \\
&\geq \frac{1}{4^\alpha}||u_m - u_n||_V^2.
\end{aligned}$$

Moreover,

$$J(u_m) + J(u_n) - 2J\left(\frac{1}{2}u_m + \frac{1}{2}u_n\right) \le \left(\Upsilon + \frac{1}{m}\right) + \left(\Upsilon + \frac{1}{n}\right) - 2\Upsilon = \frac{1}{m} + \frac{1}{n}.$$

Hence,

$$||u_m - u_n||_V^2 \le \frac{4}{\alpha}\left(\frac{1}{m} + \frac{1}{n}\right) \to 0$$

for $m, n \to 0$. Thus we have shown that $\{u_n\}$ is Cauchy in V. Thus there exists a $u \in V$ such that $u_n \to u \in V$. Thus

$$|L(u_n - u)| \le \Lambda||u_n - u||_v \to 0$$

as $n \to \infty$. Similarly,

$$\begin{aligned} |a(u_n, u_n) - a(u,u)| &= |a(u_n, u_n - u) + a(u_n - u, u)| \\ &\le |a(u_n, u_n - u)| = |a(u_n - u, u)| \\ &\le M||u_n - u||_V(||u||_v + ||u||_v) \\ &\le M||u - u_n||_V(2||u||_v + ||u_n - u||_V) \\ &\implies |a(u_n, u_n) - a(u,u)| \to 0. \end{aligned}$$

So J is a continuous function on V since $J(u_n) \to J(U)$ for $n \to \infty$. Thus,

$$J(u) = \inf_{v \in V} J(v)$$

and thus u satisfies the minimization problem. For uniqueness, suppose there exists another $\hat{u} \in V$ such taht $J(\hat{u}) \le J(v)$ for all $v \in V$ with $\hat{u} \ne u$. By definition,

$$a(u - \hat{u}, v) = 0, \quad v = u - \hat{u} \implies a(u - \hat{u}, u - \hat{u}) \le \alpha||u - \hat{u}||^2 \le a(u - \hat{u}, u - \hat{u}) = 0 \implies u = \hat{u}.$$

Observe that in the proof above, we appealed to Cauchy sequences, which require that the underlying space V is complete (V is a Hilbert space).

As a corollary, let V be a linear functional in V. Then there exists a unique $u \in V$ with

$$(u, v)_V = L(v) \quad \forall v \in V.$$

That is to say, every v has a single $u \in V$ associated with it. We call this the Riesz representation. It also follows that

$$\sup_{\substack{v \in V \\ v \ne 0}} \frac{|L(v)|}{||v||_v} = ||u||_V.$$

To show this, we want to show that $a(u, v) = (u, v)_V$ satisfies the following conditions:

- (1) Continuity: $|a(u, v)| = |(u, v)_V| \le ||u||_V||v||_V$.
- (2) Coercivity: $a(v, v)| = (v, v)_V = ||v||_V^2$, that is, $\alpha = 1$ from above.

From the Riesz representation above with $u = v$,

$$||u||_V^2 = (u, v)_V = L(u) \implies ||u||_v = \frac{|L(u)||}{||u||_V}$$

with the $u = 0$ result above being the trivial case. Then

$$\begin{aligned} ||u||_V = \frac{|L(u)|}{||u||_V} &\le \sup_{\substack{v \in V \\ v \ne 0}} \frac{|L(v)|}{||v||_v} \\ &\implies \sup_{\substack{v \in V \\ v \ne 0}} \frac{|L(v)|}{||v||_v} \le \sup_{\substack{v \in V \\ v \ne 0}} \frac{||u||_V||v||_V}{||v||_V} = ||u||_V. \end{aligned}$$

But the above assumes that $a(\cdot,\cdot)$ is symmetric. We are interested in considering whether or not this condition can be removed.

So, we want to show that there exists a unique $u \in V$ such that $a(u,v) = L(v)$ for all $v \in V$ under all of the previous assumptions but without symmetry of the bilinear form. Let $u \in V$ be fixed. Define a linear function

$$v \mapsto a(u,v).$$

We claim $a(u,\cdot)$ is linear and continuous. Observe that

$$\begin{aligned} &a(u,\mu v + w) = \mu a(u,v) + a(u,w) \\ &|a(u,v)| \le M||u||_v||v||_V. \end{aligned}$$

By the Riesz representation theorem, there exists a unique $\hat{u} \in V$ (this is a representer of v from the mapping defined above) such that

$$(\hat{u},v)_V = a(u,v) \quad \forall \in V.$$

The fact that this $\hat{u}$ is unique implies that there is a new mapping

$$u \mapsto \hat{u} \in A(u)$$

and if A is the scalar product, then this mapping is the identity. A is a linear mapping: if $u_1, u_2 \in V$, then

$$A(u_1+u_2,v) = a(u_1+u_2,v) = a(u_1,v) + a(u_2,v) = (A(u_1),v)_V + (A(u_2),v)_V = (A(u_1)+A(u_2),v)_V$$

which happens if and only if $A(u_1+u_2) = A(u_1) + A(u_2)$. Additionally,

$$\begin{aligned} (A(\mu u),v)_V = a(\mu u,v) &= \mu a(u,v) \\ &= \mu(A(u),v)_V \\ &= A(\mu u) = \mu A(u). \end{aligned}$$

Moreover, it is a bounded functional:

$$\begin{aligned} ||A(v)||_V^2 &= a(u,A(u)) \\ &\le M||u||||v||||A(u)|| \\ &\le ||A(u)||_V \le M||u||_V, \end{aligned}$$

establishing that $A(\cdot)$ is bounded. Use coercivity:

$$(A(u),u)_V = a(u,u) \ge \alpha||v||_V^2$$

from which it follows that

$$\alpha||v||_V^2 \le ||A(u)||_V||u||_V \implies \alpha||u||_V \le ||A(u)||_V$$

holds for all $v \in V$. Now consider $p > 0$ which is to be determined. The first mapping yields

$$v \mapsto (u^n,v)_V + p\{L(v) - a(u^n,v)$$

for a given $u \in V$. We want to show that this has a fixed point. By Riesz, there exists a $u^{n+1} \in V$ such that

$$(u^{n+1},v)_V = (u^n,v)_V + p\{L(u) - a(u^{n+1},v)\}$$

meaning there is a map $T : u^n \to u^{n+1}$ that is non-linear. A fixed point satisfies the variational equation. Now suppose that T has a fixed point. There exists a $u \in V$ such that $Tu = u$:

$$(Tu,v)_V = (u,v)_V + p(L(v) - a(u,v)) \iff a(u,v) = L(v).$$

To show that T has a fixed point, it suffices to show that it is contractive by the Banach fixed point theorem. Specifically, that therer exists a $\mu \in (0,1)$ such that

$$||T(u) - T(\hat{u})||_v \le \mu||u - \hat{u}||_V.$$

By definition,

$$(T(u), v)_V = (u, v)_V + p(L(v) - a(u, v))$$

and similarly for $T(\hat{u})$. Subtracting and rearranging,

$$\begin{aligned}(T(u) - T(\hat{u}), v)_V = (u - \hat{u}, v)_V - pa(u - \hat{u}, V) &= (u - \hat{u}, v)_V - p(A(u - \hat[u, v)_V \\ &\implies T(u) - T(\hat{u}) = (u - \hat{u}) - pA(u - \hat{u}) \\ &\implies ||T(u) - T(\hat{u})||_V^2 = ||u - \hat{u}||_V^2 - 2p(A(u - \hat{u}), u) + p^2||A(u - \hat{u})||^2 \\ &= ||u - \hat{u}||^2 - 2pa(u - \hat{u}, u - \hat{u}) + p^2||A(u - \hat{u})||^2 \\ &\leq ||u - \hat{u}||_V^2 - 2p\alpha||u - \hat{u}||_V^2 + p^2M^2||u - \hat{u}||_V^2 \\ &\implies ||T(u) - T(\hat{U})||_V^2 \leq (1 - 2\alpha p + p^2M^2)||u - \hat{u}||_V^2 \\ &= \left(1 - \frac{\alpha^2}{M^2}\right)||u - \hat{u}||_V^2\end{aligned}$$

with the choice of p given by $2pM^2 = 2\alpha \implies p = \alpha/M^2$. This establishes that T is contractive and hence by Banach has a unique fixed point.

We now state the general Lax-Milgram lemma. Let V be a Hilbert space with scalar product $(\cdot, \cdot)_V$. Let $L : V \to \mathbb{R}$ be bounded and lienar and $a : V \times V \to \mathbb{R}$ be bilinear and continuous, meaning

$$||a(u, v)|| \leq \mu||u||_V||v||_V$$

and coercive, meaning

$$a(v, v) \geq \alpha||v||_V^2 \quad \forall v \in V$$

then there exist a unique $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V.$$

Moreover, the solution is bounded by

$$||u||_V \leq \frac{\Lambda}{\alpha}.$$

To prove this, choose $v = u$ in the previous discussion, so

$$\alpha||u||_V^2 \leq a(u, v) = L(u) \leq \Lambda||u||_V.$$

We now consider some application examples of the above theorem. Consider the string, where $V = H_0^1(0, L)$,

$$\begin{aligned}a(u, v) &= \int_0^L u'v' \\ L(v) &= \int_0^L fv + pv(x_p).\end{aligned}$$

Observe that V is complete with the scalar product

$$(u, v)_{H^1} = (u, v) + (u', v")$$

and $L(v)$ is linear. Moreover,

$$\begin{aligned}|L(v)| &\leq |(f, v)| + |p| \cdot |_V(x_p)| \\ &\leq ||f|| \cdot ||v|| + ? \\ &\leq ||p|| \cdot ||v||_{H^1}.\end{aligned}$$

So

$$|v(x_p)|^2 \leq \left|\int_0^{x_p} v'(s)ds\right|^2 \leq \int_0^{x_p} ds' \int_0^{x_p} v'(s)^2 ds \leq x_p \int_0^L v'(s)^2 ds \leq x_p||v||_{H^1}^2.$$

This is the Sobolev inequality

$$|v(x_p)| \le \sqrt{x_p}||v||_{H^1} \le ||f|| \cdot ||v|| + ||p||\sqrt{x_p}||v||_{H^1} = (||f|| + |p|\sqrt{x_p})||v||_{H^1}.$$

We call

$$\Lambda = (||f|| + |p|\sqrt{x_p}).$$

Also,

$$\begin{aligned} |a(u,v)| = \left|\int_0^L Tu'v'\right| &\le \sqrt{\int_0^L T(u')^2} \cdot \sqrt{\int_0^L T(v')} \\ &\le \max_{0\le s\le L} T(s)||u'|| \cdot ||v'|| \\ &\le \max_{0\le s\le L} T(s)||u||_{H^1}||v||_{H^1}. \end{aligned}$$

To show coercivity, we want

$$\alpha(||v||^2 + ||v'||^2) \le a(v,v) = \int_0^L T(v')^2 ds.$$

Observe that

$$\begin{aligned} v(x) = \int_0^x v'(s)ds \implies ||v(x)||^2 &\le \left(\int_0^x v'(s)ds\right)^2 \\ &\le \int_0^x ds' \cdot \int_0^x v'(s)^2 ds \\ &\le x \cdot \int_0^L v'(s)^2 ds \\ &= x||v'||^2 \\ &\implies ||v||^2 \le \frac{L^2}{2}||v'||^2 \\ &\implies ||v|| \le \frac{L}{\sqrt{2}}||v'|| \end{aligned}$$

by passing through the L_2 norm. So

$$||v||^2 + ||v'||^2 \le \left(1 + \frac{L^2}{2}\right)||v'||^2,$$

which is another Sobolev inequality. Also

$$\begin{aligned} a(v,v) = \int_0^L Tv'(s)^2 ds &\ge \min_{0\le x\le L} T(x) \int_0^L v'(s)^2 ds \\ &= \min_{0\le x\le L} T(x)||v'||^2 \\ &\ge \min_{0\le x\le L} T(x)\frac{1}{1+\frac{L^2}{2}}||v||_{H^1}^2. \end{aligned}$$

where the $\alpha > 0$ is the constant term in the last expression. From Lax-Milgram, the string problem admits a unique solution.

For the Euler-Bernoulli beam, $V = H_0^2$ and

$$\begin{aligned} (u,v)_V &= (u,v) + (u',v') + (u'',v'') \\ a(u,v) &= \int_0^L EIu''v'' \\ L(v) &= \int_0^L fv + pv(x_p) + Qv'(x_Q). \end{aligned}$$

L is linear. To show that L is bounded, consider

$$|L(v)| \leq ||f|| \cdot ||v|| + |p| \cdot |v(x_p)| + |Q| \cdot |v'(x_Q)|.$$

As before, Sobolever yields

$$|v(x_p)| \leq \sqrt{x_p}||v||_{H^1} \leq \sqrt{x_p}||v||_{H^2}$$

and

$$|v'(x_Q)| \leq \sqrt{x_Q}||v'||_{H^1} \leq \sqrt{x_Q}||v'||_{H^2}.$$

So L is bounded. Finally,

$$\begin{aligned}|a(u,v)| &= \left|\int_0^l EIu''v''\right| \\ &\leq \sqrt{\int_0^L EI(u'')^2} \cdot \sqrt{\int_0^L EI(v'')^2} \\ &\leq \max EI||u''|| \cdot ||v''|| \leq \cdots ||u||_{H^2} \cdot ||v||_{H^2}.\end{aligned}$$

We need

$$\alpha(||v||^2 + ||v'||^2 + ||v''||^2) \leq a(v,v) = \int_0^L EI(u'')^2$$

and

$$||v||^2 \leq \frac{L^2}{2}||v'||^2$$

and

$$||v'||^2 \leq \frac{L^2}{2}||v''||^2.$$

Thus

$$||v||^2 + ||v'||^2 + ||v''||^2 \leq \left(1 + \frac{L^2}{2}\right)||v'||^2 + ||v''||^2 \leq \left(1 + \frac{L^2}{2}\right)$$

I'm not too sure how this proof concludes...

Lecture 7 [3/9]

Recall that the Dirichlet problem is given by

$$\begin{aligned}-\Delta u &= f \quad \Omega \subset \mathbb{R}^d \\ u &= 0 \quad \partial\Omega\end{aligned}$$

while the Neumann problem is given by

$$\begin{aligned}-\Delta u &= f \quad \Omega \\ \frac{\partial u}{\partial \ltimes} &= g\end{aligned}$$

where $\ltimes$ is the exterior normal to the boundary $\partial\Omega$.

For example the string problem in $\mathbb{R}$ reduces to $-u'' = f$ in (a,b) with $u(a) = u(b) = 0$. By integration by parts,

$$\int_\Omega fv = \int_\Omega -\Delta u \cdot v = \int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v.$$

The variational forms are: (1) find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall H_0^1(\Omega)$$

where
$$a(u,v)=\int_\Omega \nabla\cdot u\nabla\cdot v$$
and
$$H_0^1(\Omega)=\{v:\Omega\to\mathbb{R}, v,\nabla v\in L^2, v=0 \text{ on } \partial\Omega\}.$$
and similarly for $H^1(\Omega)$ where we don't impose the boundary condition. The variational problems correspond to the Dirichlet and Neumann boundary conditions.

To continue our studies, it is important to talk more about Sobolev spaces. Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain. For $m\in\mathbb{N}$, define the Sobolev norm
$$||v||^2_{H^m(\Omega)}=||v||^2_{L^2}+||Dv||^2_{L^2}+\cdots+||D^m v||^2_{L^2}$$
where, for example, in the 2D version,
$$Dv=\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right),\quad D^2v=\left(\frac{\partial^2 v}{\partial x^2},\frac{\partial^2 v}{\partial x\partial y},\frac{\partial^2 v}{\partial y^2}\right),\cdots,D^m v=\left(\frac{\partial^m v}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\right)$$
where $0\le\alpha_1,\alpha_2$ and $\alpha_1+\alpha_2=m$. Let $C^\infty(\Omega)$ be the space of smooth functions on Ω. Then define H^m as
$$v\in H^m(\Omega)\iff\exists\{v_n\}\subset C^\infty(\Omega):||v_i-v_j||_{H^m}\to 0, n\to\infty$$
which is to say that $\{v_n\}$ is a Cauchy sequence in the H^m norm. We have thus constructed a complete space by definition.

We now consider whether H^m is well-defined. Let $\alpha=1$ and $v\in H^m(0,1)$. Is v well-defined point-wise? The answer depends on what m is!

- Case 1: $m=0$. Choose the sequence
$$v_n(x)=x^n\in C^\infty(0,1)\quad\forall n$$
then
$$\begin{aligned}||v_n-v_k||^2_{H^0}&=||v_n-v_k||^2_{L^2}\\&=\int_0^1(x^n-x^k)^2dx\\&\le\frac{1}{2n+1}+\frac{2}{n+k+1}+\frac{2}{k+1}\\&\to 0,\quad n,k\to\infty\end{aligned}$$
So by the definition we chose for H^m, there exists $v\in H^0$ such that $v_n\to v\in H^0$ and pointwise
$$v_n(x)\to\begin{cases}1 & x=1\\0\end{cases}$$
so it would make sense that that is what $v(x)$ is. On the other hand,
$$||v_n||^2_{H^0}=\int_0^1 x^{2n}dx=\frac{1}{2n+1}\to 0, n\to\infty$$
so $||v_n-0||^2_{H^0}\to 0$, so what, is this the zero function in the limit? Does it really make sense to define convergence pointwise here? Well it seems like this cannot distinguish between pointwise discrepencies. But why do we care about pointwise values anyway? The reason for that is that boundary conditions rely on pointwise values!

- $m = 1$... Okay so we have problems with $m = 0$, but what happens with $m = 1$? Observe that $v_n(x) = x^n$ is no longer Cauchy in H^1. But do we still have well-defined pointwise values? Define the tracemap $\gamma_0 v(x) :=$ "$v(x)$". The claim is that there exists a $C < \infty$ such that $|\gamma_0(x)| \leq C||v||_{H^1}$ for all $v \in H^1$. To show this choose a $v \in H^1$, then there exists $v_n \in C^\infty$ such that $||v_n - v||_{H^1} \to 0$ as $n \to \infty$. Now

$$v_n(x) = v_n(y) + \int_y^x v_n'(s)ds$$
$$\implies |v_n(x)| \leq |v_n(y)| + |x-y|\sqrt{\int_y^x v_n'(s)^2 ds}$$
$$\implies \frac{1}{2}|v_n(x)|^2 \leq |v_n(y)^2 + |x-y|||v_n'||^2_{L^2}$$

where $|x-y| \leq b-a$. Integrate over $y \in (a,b)$, so

$$\frac{1}{2}(b-a)|v_n(x)|^2 \leq ||v_n||^2 + (b-a)^2||v_n'||^2 \leq \max\{1,(b-a)^2\}||v_n||^2_{H_1} \implies |v_n(x) \leq C||v_n||^2_{H^1}$$

for all $v_n \in C^\infty$. This likewise applies to $v_n - v_k$ so

$$|v_n(x) - v_k(x)|^2 \leq C||v_n - v_k||^2_{H^1}$$

so it's Cauchy in H^1. Note that if th RHS is Cauchy in H^1, then the LHS is Cauchy as well... but in what? Well $LHS \in \mathbb{R}$ so $\{v_n(x)\}$ is Cauchy in $\mathbb{R}$ so by completeness of $\mathbb{R}$, there exists some limit $\gamma_0(x) \in \mathbb{R}$ to which the sequence converges. To explain the notation further, γ_0 defines a map from $v \in H^1$ to $\gamma_0 v(x)$ which is associated with the pointwise value of v at x. Anyway this gives us

$$\begin{aligned}|\gamma_0 v(x)| &\leq |\gamma_0 v(x) - v_n(x)| + |v_n(x)| \\ &\leq |\gamma_0 v(x) - v_n(x) + C||v_n - v||_{H^1} + C||v||_{H^1} \\ &\to |\gamma_0 v(x)| \leq C||v||_{H^1} \quad \forall v \in H^1.\end{aligned}$$

This is known as the trace inequality. The splitting of terms trick above forms a density argument that will be useful later on as well.

With the reasoning above, let's return to the string problem. The variational interpretation of the boundary conditions is $u(a) = u(b) = 0$ is instead $\gamma_0 u(a) = \gamma_0 u(b) = 0$. This can be thought of an equivalence class that ignores jumps on a set of measure 0. So this gives that poitnwise loads are well-defined in H^1 and

$$L(v) = pv(p) \sim p\gamma_0 v(p)$$

where the inequality gives $|L(v)| \leq C|p|||v||_{H^1}$. So what about torques to the string? Torque corresponds to $Mv'(x_n)$ in $L(v)$. This is bounded only if we give an interpretation of $v'(x_n)$ as $\gamma_1 v(x_n)$ satisfying

$$|\gamma_1 v(x_m)| \leq C||v||_{H^1} \quad v \in H^1$$

However, this is not possible, since if this were, then pointwise values would make sense in H^0.

What about higher dimensions? So we need at least H^1, so $m \geq 2$ but what about the dimension $d \geq 2$? Does pointwise evaluation make sense in H^1 on $\mathbb{R}^d$ where $d \geq 2$? No. Consider $u(x) = \log\log\frac{1}{|x|}$. This is in H^1 but $u(x)$ is not continuous at $x = 0$ so γ_0 cannot be defined properly, so there is no trace operator possible at $x = 0$. THe problem here is that we've dropped 2 dimensions... Without pointwise meaning, we cannot handle boundary conditions, so how about we weaken the interpretation of what BC mean. Let $u = 0$ in L^2 because it's weaker to say equivalence in $L^2(\partial\Omega)$. Because crazy things came in from our definition of H^m, we need to weaken the interpretation. Let's drop 1 dimension.

We can define traces in higher dimensions. Let $\Omega \subset \mathbb{R}^d$ and $\partial\Omega \subset \mathbb{R}^{d-1}$. Can we make sense of the restriction of $u \in H^1(\Omega)$ to $\partial\Omega$ for the BCs?

Break up the boundary into little bits consisting of wedges w_m, arcs Γ_m and points x_m. Define a smooth vector field p_0 on a wedge w_0 such that $n \cdot p_0 = 0$ on $\partial w_0/\Gamma_0$ (parallel to edge) and $n \cdot p_0 \geq \alpha > 0$ on Γ_0 (doesn't have to be perpendicular, just non-parallel is enough. For example $p_0(x) = x - x_0$.

Let $v \in H^1(\Omega)$ then there exists $\{v_n\} \subset C^\infty$ such that $||v_n - v_m||_{H^1}$ is Cauchy and $v_n \to v$ in H^1. From this, it follows that

$$\begin{aligned}
\alpha||v_n||^2_{L^2(\Gamma_0)} &\leq \int_{\Gamma_0} n \cdot p_o v_n^2(s) ds \\
&= \int_{\partial w_o} n \cdot p_o v_n^2(s) ds \\
&= \int_{w_o} \div(p_o v_n^2) dx \\
&= \int_{w_o} (\div p_o) v_n^2 + \int_{w_o} 2 v_n p_o \cdot \nabla v_n dx \\
&\leq d||v_n||^2_{w_o} + 2||p_o||_\infty \int_{w_o} |v_n| \cdot |\nabla v_n| dx \\
&\leq \alpha||v_n||^2_{w_o} + 2||p_o||_\infty ||v_n||_{w_o} |||\nabla v_n||_{w_o} \\
&\leq C\{||v_n||^2_{w_o} + ||\nabla v_n||^2_{w_o} \\
&= C||v_n||^2_{H^1(w_o)}
\end{aligned}$$

where we have defined $d = \div p_o$ and used the trivial inequality $ab \leq \frac{1}{2}(a^2 + b^2)$. The same argument applies to all arcs $\Gamma_0, \Gamma_1, \ldots$ so summing across everything yields

$$\alpha||v_n||^2_{L^2(\partial\Omega)} \leq C||v_n||^2_{H^1(\Omega)}.$$

Time to construct the trace. The above inequality applies to $v_n - v_k$ so

$$||v_n - v_k||^2_{L^2(\partial\Omega)} \leq \frac{C}{\alpha}||v_n - v_k||^2_{H^1(\Omega)} \to 0$$

so $\{v_n\}$ is Cauchy in L^2 as $n, k \to \infty$ as well. By completeness of $L^2(\partial\Omega)$, there exists some $\gamma_0 v \in L^2(\partial\Omega)$ such that $||v_n - \gamma_0 v||_{L^2(\partial\Omega)} \to 0$. The trick here is to leverage the completeness of subspaces. Now we show boundedness as before in the $d = 1$ case. Observe that by another density argument,

$$\begin{aligned}
||\gamma_0 v||_{L^2(\Omega)} &\leq ||\gamma v - v_n||_{L^2(\partial\Omega)} + ||v_n||_{L^2(\partial\Omega)} \\
&\leq ||\gamma_0 v - v_n||_{L^2(\partial\Omega)} + C||v_n||_{H^1} \\
&\leq ||\gamma_0 v - v_n||_{L^2(\partial\Omega)} + C||v_n - v||_{H^1(\Omega)} + C||v||_{H^1(\Omega)} \\
&\to C||v||_{H^1(\Omega)}
\end{aligned}$$

for $n \to \infty$. Thus we have defined a map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ which is continuous and linear. This is the $L^2(\partial\Omega)$ trace map in $\mathbb{R}^d$.

So we can say that $\gamma_0 u$ is 0 on $\partial\Omega$. But we cannot say $u = 0$ on $\partial\Omega$. That is, the well-defined weaker interpretation is true, but the other is too strict and not well-defined. Interpret boundary conditions for Dirichlet as

$$-\Delta u = f, \quad \gamma_0 u = 0.$$

But what exactly is the space of functions $\gamma_0 v$? The range of γ_0 is a subset of $L^2(\partial\Omega) = H^0(\partial\Omega)$ but it is not a subset of $H^1(\partial\Omega)$ because we lost a derivative. But is there anything in between? Yes, in fact, the range is exactly $H^{1/2}(\partial\Omega)$. This means that the correct formulation of the Dirichlet problem is

$$u \in H_0^1 : a(u, v) = (f, v) \quad \forall v \in H_0^1$$

where

$$H_0^1 = \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ on } \partial\Omega\}.$$

Is H_0^1 complete? H^1 is complete and $H_0^1 \subset H^1$ but is H_0^1 also complete? Well it suffices to show that H_0^1 is closed because a closed subspace of a complete space is complete. Pro gamer tip: never show a space is complete. Always try first via a closed subspace of another complete space!

So for the proof. Let $v_n \in H_0^1 \iff \gamma_0 v_n = 0$ in $L^2(\partial\Omega)$ we want to show $\gamma_0 v = 0$. Well

$$\begin{aligned}||\gamma_0 v_n||_{L^2(\partial\Omega)} &\leq ||\gamma_0 v - \gamma v_n||_{L^2(\partial\Omega)} + ||\gamma_0 v_n||_{L^2(\partial\Omega)} \\ &\leq C||\gamma_0 v - \gamma v_n||_{H^1(\Omega)} \\ &\to 0\end{aligned}$$

so $||\gamma_0 v||_{L^2(\partial\Omega)} = 0 \iff \gamma v = 0$ on $L^2(\partial\Omega)$ and thus H_0^1 is complete!

By the reasoning above, we want to use Lax-Milgram to show existence of a solution in H_0^1. We have that H_0^1 is complete. We also have continuity of the operator:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \leq ||\nabla u|| \cdot ||\nabla v|| \leq ||u||_{H^1}||v||_{H^1}.$$

And a is coercive: we want to show there exists $\alpha > 0$ such that $a(v,v) \geq \alpha||v||_{H^1}^2 \iff \alpha(||v||^2 + ||\nabla v||^2) \leq ||\nabla v||^2$. This is true if there exists a C such that $||v||^2 \leq C||\nabla v||^2$ for all $v \in H_0^1(\Omega)$ which leads to the following.

Lemma due to Poincare. Let $v \in H_0^1$ be given. Then there exists a $v_n \in C^\infty$ such that $\gamma_0 v_k = 0$ and $v_n \to v$ in H^1. Since $\gamma_0 v_n = v_n$ when $v_n \in C^\infty$, we can extend v_n by 0 to a box containing Ω. That is $\Omega = (a,b)^d$. Now

$$\begin{aligned}v_n(x,y) = v_n(x,a) + \int_a^y \frac{\partial v_n}{\partial y}(x,t)dt \\ \implies |v_n(x,y)|^2 \leq |y-a|\int_a^y \frac{\partial v_n}{\partial y}(x,t)^2 dt \\ \leq |b-a|\int_a^b \frac{\partial v_n}{\partial y}(x,t)^2 dt.\end{aligned}$$

Integrating over y,

$$\int_a^b |v_n(x,y)|^2 dx \leq |b-a|^2 \int_a^b \frac{\partial v_n}{\partial y}(x,t)^2 dt.$$

So integrating over x,

$$||v_n||_\Omega^2 \leq |b-a|^2 \left|\left|\frac{\partial v_n}{\partial y}\right|\right|_\Omega^2 \leq |b-a|^2||\nabla v_n||_\Omega^2.$$

Then by another density argument,

$$\begin{aligned}||v||_\Omega &\leq ||v - v_n||_\Omega + ||v_n||_\Omega \\ &\leq ||v-v_n||_{H^1(\Omega)} + C||\nabla v_n||_\Omega \\ &\leq ||v-v_n||_{H^1(\Omega)} + C||\nabla v_n - \nabla v||_\Omega + L||\nabla v||_\Omega \\ &= C||\nabla v||_\Omega.\end{aligned}$$

Having proved that lemma, we show coercitivity...

$$\begin{aligned}||v||_{H^1}^2 &= ||v||_{L^2}^2 + ||\nabla v||_{L^2}^2 \\ &\leq (1+c_))||\nabla v||_{L^2}^2 \\ &\implies \frac{1}{1+c_p}||v||_{H^1}^2 \leq ||\nabla v||_{L^2}^2 = a(v,v).\end{aligned}$$

So choosing $\alpha = \frac{1}{1+c_p}$, we are done. So

$$|L(v)| = |(f,v)| \leq ||f|| \cdot ||v|| \leq ||f||_{H^1} \cdot ||v||_{H^1}.$$

So we can use Lax-Milgram.

Lecture 8 [3/16]

We continue the duet between the Dirichlet and Neumann problems. Recall from earlier that Dirichlet gives the following variational problem

$$u \in H_0^1 : a(u,v) = L(v) \quad \forall v \in H_0^1$$

and the Dirichlet boundary condition implied that the trace $\gamma_0 v = 0$ on $\partial\Omega$ giving

$$H_0^1 = \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ on } \partial\Omega\}.$$

We then employed the Poincare inequality to show coercivity:

$$||v||_{L^2(\Omega)} \leq C||\nabla v||_{L^2} \quad \forall v \in H_0^1.$$

Then by Lax-Milgram, we concluded that there exists a unique solution.

We now switch to the Neumann problem. The variational form is given by

$$u \in H^1(\Omega) : (f,v) = (-\Delta u, v) = (\nabla u, \nabla v) - \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds$$

where $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$ is the Neumann condition. There needs to be an argument for applying the divergence theorem above and that ties into the $H^{-1/2}$ Sobolev space, but that is omitted since that is not the main thrust of this lecture. The above is equivalent to

$$u \in H^1(\Omega) : a(u,v) = L(v); \quad L(v) = (f,v) + \int_{\partial\Omega} g v ds, \ a(u,v) = (\nabla u, \nabla v).$$

The question we are going to pursue is whether or not this variational formulation is a good one. That is, given the above, is Lax-Milgram applicable? Also, note that Lax-Milgram gives a sufficient condition – not a necessary condition – for the uniqueness!

It's easy to check completeness, linearity, and continuity, but the issue is with coercivity. Observe that we want to find $\alpha > 0$ such that

$$a(v,v) = ||\nabla v||^2 \geq \alpha ||v||_{H^1}^2 \quad \forall v \in H^1(\Omega)$$

but this is not true when v is constant. Thus, the operator is not coercive by counterexample. Thus, we conclude that there are problems with the variational formulation. We can investigate this issue further. Suppose there exists a u that satisfies the variational form. Then $u + c$ also satisfies it:

$$a(u+c, v) = a(u,v) + a(c,v) = a(u,v) = L(v) \quad \forall v \in H^1(\Omega)$$

meaning solutions are not unique. How about existence? Choose v to be a constant. Then

$$a(u,c) = 0 = L(c) = cL(1) \implies L(1) = \int_\Omega f + \int_{\partial\Omega} g$$

which gives a necessary condition for existence of the solution. Note that this condition was not given as part of the problem. However, this has a very important physical interpretation. Namely, this is a steady-state problem. Consider the Fourier heat law with positive flux f per unit area and g being the rate of heat diffusion out of Ω. Then $\frac{\partial u}{\partial n} = g$ and for the heat flux, Fourier's heat law gives

$$q = -\nabla u, \quad n \cdot q = -n \cdot \nabla u = -\frac{\partial u}{\partial n}.$$

Thus the steady-state condition means that the rate of heat increase on Ω is equal to the heat flow across $\partial\Omega$:

$$\int_\Omega f + \int_{\partial\Omega} g = 0.$$

This is heat balance and gives a "compatibility condition" on the data. In summary, if a solution exists, it is not unique. And a solution can only exist if the data satisfy the compatibility condition. This can be likened to the following problem from basic linear algebra:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}; \quad A\vec{x} = \vec{b}$$

The solution is not unique since if $\vec{x}$ exists, then $\vec{x}+\vec{c}$ also satisfies the system while the existence of a solution implies that $2b_1 + b_2 = 0$. These sorts of problems are apparently part of the greater Fredholm theory.

So, having seen that Lax-Milgram is not applicable in the problem above, we conclude that the space in which we are solving, $H^1(\Omega)$ is too large as it admits functions such that $a(v,v) = 0$ when $v \neq 0$. This can be understood in the context of potentials and pressures and other physical quantities being determined up to a constant. To remedy this, suppose we worked on the subspace

$$V \subset H^1(\Omega) = \{v \in H^1(\Omega) : v(0) = 0\},$$

which doesn't make sense since point-wise values of functions in H^1 do not make sense, for example, with $\log\log\frac{1}{r}$. Instead, we smear this value by considering the average value of the function. Define

$$V = \{v \in H^1(\Omega) : \int_\Omega v = 0\}$$

which is well-defined. This fixes the problem with constants, but is the mapping $G : v \mapsto \int_\Omega v \in \mathbb{R}$ continuous? We want to show that there exists a constant C_G such that $|G(v)| \leq C_g||v||_{H^1(\Omega)}$ for every $v \in H^1(\Omega)$. This is true if and only if

$$\left|\int_\Omega v\right| \leq C_G||v||_{H^1}$$

assuming a finite domain. But

$$\left|\int_\Omega v\right| = |(v,1)| \leq ||v|| \cdot ||1|| \leq ||v||_H^1 \cdot ||1||,$$

so choosing $C_G = ||1|| = \sqrt{|\Omega|}$, this is true if Ω is bounded. With this V, consider the variational problem

$$u \in V : a(u,v) = L(v) \quad \forall v \in V.$$

Since G is continuous, V is closed in $H^1(\Omega)$ implying completeness. Continuity:

$$|a(u,v)| \leq ||\nabla u|| \cdot ||\nabla v|| \leq ||u||_{H^1}||v||_{H^1},$$

and since $u, v \in H^1$, it follows for a subspace. Also

$$|L(v)| \leq (||f|| + C_k||g||)||v||_{H^1}.$$

For coercivity, we need to show that there exists $\alpha > 0$ such that $a(v,v) \geq \alpha||v||_{H^1}$ for all $v \in V$. This is true if and only if

$$||\nabla v||^2 \geq \alpha(||\nabla v||_{L^2}^2 + ||v||_{L^2}^2) \geq \alpha||v||^2.$$

So we need to show that

$$||v||^2 \leq C||\nabla v||^2.$$

This is a lemma due to Poincare.

Let Ω be a square of side length $D > 0$. Then there exists $C_p > 0$ such that

$$||v - \bar{v}_\Omega|| \leq C_p||\nabla v||$$

where

$$\bar{v}_\Omega = \frac{1}{|\Omega|}\int_\Omega v$$

is the average across Ω. Take $P(p_1, p_2)$ and $Q(q_1, q_2) \in \Omega$ and let $R(q_1, p_2) \in \Omega$. Then

$$u(P) - u(Q) = (u(P) - u(R)) + (u(R) - u(Q))$$
$$u(P) - U(R) = u(p_1, p_2) - u(q_1, p_2) = \int_{q_1}^{p_1} \frac{\partial u}{\partial x}(x, p_2)dx.$$

It follows that

$$|u(P) - u(R)|^2 \le D \int_a^b \frac{\partial u}{\partial x}(x, p_2)^2 dx$$
$$|u(R) - u(Q)|^2 \le D \int_a^b \frac{\partial u}{\partial y}(q_1, y)^2 dy$$

Recall that

$$a \le b + c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} b \\ c \end{pmatrix} \implies \frac{1}{2}a^2 \le b^2 + c^2$$

so

$$\frac{1}{2}|u(P) - u(Q)|^2 \le D \left[\int_a^b dx \frac{\partial u}{\partial x}(x, p_2)^2 + \int_a^b dy \frac{\partial u}{\partial y}(q_1, y)^2. \right]$$

Integrating over all $P \in \Omega$,

$$\frac{1}{2} \int_\Omega dP|u(P) - u(Q)|^2 \le D \left\{ D \int_a^b dx \int_a^b dp_2 \frac{\partial u}{\partial x}(x, p_2)^2 + D^2 \int_a^b \frac{\partial u}{\partial y}(q_1, y))^2 dy \right\}$$

Integrating over all $Q \in \Omega$,

$$\frac{1}{2} \int_\Omega dP \int_\Omega dQ(u(P) - u(Q))^2 \le D^4 ||\nabla u||_\Omega^2$$

while the LHS is

$$\frac{1}{2}D^2 \int_\Omega dPu(P)^2 + \frac{1}{2}D^2 \int_\Omega dQu(Q)^2 - \int_\Omega dPu(P) \int_\Omega dQu(Q) = D^2||u||_\Omega^2 - D^2\bar{u}_\Omega \cdot D^2\bar{u}_\Omega = D^2||u||_\Omega^2 - (D^2\bar{u}_\Omega)^2.$$

It follows that

$$\begin{aligned} D^2||u - \bar{u}_\Omega||^2 &= D^2[||u||_\Omega^2 - 2(\bar{u}_\Omega, u) + ||\bar{u}_\Omega||^2] \\ &= D^2[||u||_\Omega^2 - 2\bar{u}_\Omega(1, u) + \bar{u}_\Omega^2 D^2] \\ &= D^2||u||_\Omega^2 \cdot \bar{u}_\Omega^2 D^4 \le D^4||\nabla u||_\Omega^2 \end{aligned}$$

The immediate consequence is that $u \in V \iff \bar{u}_\Omega = 0$ gives $||v||_\Omega^2 \le C_P^2||\nabla v||^2$ from which it follows that

$$||v||^2 = ||v||^2 + ||\nabla v||^2 \le (1 + C_P)||\nabla v||^2 = (1 + C_P)a(v, v).$$

Coercivity for the problem above follows immediately with $\alpha = \frac{1}{1+C_P}$. Thus we get the following theorem for well-posedness of the Neumann problem. Suppose $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ satisfy

$$\int_\Omega f + \int_{\partial\Omega} g = 0,$$

then there exists a unique solution $u \in V = \{v \in H^1(\Omega) : \int_\Omega v dx = 0\}$ for the problem

$$(\nabla u, \nabla v) = (f, v) + \int_{\partial\Omega} gv \quad \forall v \in V$$

and depends continuously only on the data.

As a side remark, note that $V \cong H^1(\Omega)/\mathbb{R}$ is a quotient space.