APMA2570B: Numerical Solutions to PDE III

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Lecture 1 9/7:

The purpose of this course is to transition to topics of modern relevance in FEM theory. Specifically towards developing methods that are applicable to PDEs that arise from physics, rather than the basic examples from 2560. As such, the course topics are primarily focused on developments since the early 2000s, as classical FEM theory ended sometime in the '90s.

One of the main issues with classical FEM is that those schemes do not often converge to physical solutions, though they attain provable mathematical convergence. Which is to say that something fundamental is going wrong in the formulation of these schemes that makes them poorly suited to real-world application as the scheme constructs a solution correctly, but that solution is not representative of any physical process. One such example is an application of classical FEM to the Maxwell equations.

As a hint of what might be going wrong, we consider the de Rham complex

$$H^1 \xrightarrow{\text{grad}} H^1(\text{curl}) \xrightarrow{\text{curl}} H^1(\text{div}) \xrightarrow{\text{div}} \mathcal{L}^2$$

For some PDEs, solutions require more, or less, regularity than the space in which the variational form is posited. Each space in the complex there leading up to $\mathcal{L}^2$ has more or less regularity than the terminal space. Here,

$$H^1(\text{curl}) = \{v \in (\mathcal{L}^2)^3 : \nabla \times v \in (\mathcal{L}^2)^3\},$$

for instance. This conception might be useful for problems that involve conservation laws of the form

$$-\text{div}\sigma = f.$$

Specifically, to discretize that equation, the scheme must be constructed in $H^1(\text{div})$.

The purpose here is to build a mathematical theory that leads to provably convergent, stable schemes that are not ad hoc, which the literature is filled with. For decades, classical FEM has been modified ad hoc for problems to improve performance, but the point of this course is to focus on the deep mathematical theory going on behind the scenes.

Another example is the Stoke complex that serves as a guide for solving the Stokes equations. When it comes to solving mass-conserving equations like the incompressible (incomprehensible) Navier-Stokes. This involves a discretization of the equations governing the pair (u, p) in 4D and the scheme should satisfy point-wise mass conservation, balanced mass and pressure transport. Anyway all of this is to say that the schemes here are intended to solve physically-relevant problems that classical FEM was incapable of doing.

Now we move on to things of technical substance rather than just a preview. Here, we move past the Lax-Milgram theorem. As a refresher, given a problem in variational form

$$u \in X : a(u, v) = L(v) \quad \forall v \in V$$

where X is a Hilbert space, L is bounded and linear, $a : X \times X \to \mathbb{R}$ is bilinear, and is coercive/elliptic, meaning $\exists \alpha > 0 : a(v, v) \geq \alpha ||v||_X^2$ for all $v \in X$, Lax-Milgram says that there exists a unique solution of the variational problem. It's nice, but the problem is that this is rarely applicable in the real world. This isn't applicable to the Stokes equation:

$$\begin{cases} -\Delta u + \nabla p & = f \\ \nabla \cdot u & = 0 \end{cases}$$

Let $u \in H^1 \times H^1$ then

$$\begin{aligned}(f, v) &= \int_\Omega f \cdot v \\ &= \int_\Omega v(-\Delta u + \nabla p) \\ &= \sum_{i=1}^{2} \int_\Omega v_i(-\Delta u_i + \partial_{x_i} p) \\ &= \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i + \partial_{x_i} v_i \\ &= \int_\Omega \nabla u : \nabla v + \int_\Omega \nabla p \cdot v \\ &= \int_\Omega \nabla u : \nabla v - \int_\Omega p \nabla \cdot u.\end{aligned}$$

Now let $q \in \mathcal{L}^2$, then

$$0 = \int_\Omega (\nabla \cdot u) q,$$

and the weak form

$$\int_\Omega \nabla u : \nabla v - \int_\Omega p \mathrm{div} v = \int_\Omega f \cdot v = 0$$

for all $v \in H^1 \times H^1$ and for all $q \in \mathcal{L}^2$. However, superficially, Lax-Milgram isn't applicable because there are 2 equations. But of course this can be reformulated as a single equation in the unknown triple (u, p) with the (Stokes) bilinear form

$$B((u, p), (v, q)) = \int_\Omega \nabla u : \nabla v - \int_\Omega p \mathrm{div} v + \int_\Omega q \mathrm{div} u$$

and the linear form

$$L(v, q) = \int_\Omega f \cdot v.$$

Nowe we have the variational problem

$$B((u, p), (v, q)) = L(v, q) \quad \forall (v, q) \in H^1 \times H^1 \times \mathcal{L}^2.$$

The bilinear form is continuous, the linear form is linear, etc... all the conditions seem to be satisfied... except ellipticity. the LHS can be negative! So Lax-Milgram is not applicable.

Another goodie is that Lax-Milgram doesn't even apply to simple time-stepping ODEs like $u' = f$ with $u(0) = u_0$ on $(0, L)$. The variational problem for $u \in H^1(0, L)$ is

$$\int_0^L f v = \int_0^L u' v \quad \forall v \in \mathcal{L}^2.$$

The bilinear form is continuous, the linear form is continuous and linear, but Lax-Milgram is not applicable. The test space $\mathcal{L}^2$ is different from the trial space H^1 and ellipticity is again not satisfied.

An even simpler example is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

which has the variational form

$$f \cdot v = v_1 u_2 - v_2 u_1$$

and we have $B(v, v) = 0$. Big sad. This isn't elliptic either but has a unique solution.

The Stokes problem is actually similar and has a similar skew-symmetric structure to the above: the differential operator is

$$\begin{bmatrix} -\Delta & -\nabla \\ -\nabla \cdot & 0 \end{bmatrix}$$

So, we turn to the main subject of this class. Considering an operator equation $Bu = f \in Y$ where $X \neq Y$ are Hilbert spaces and $B : X \to Y$ is a linear mapping, we want to find necessary and sufficient conditions on B and X, Y such that a unique solution exists and depends continuously on the data.

Lecture 2 - 9/14:

Here we really move on past the Lax-Milgram theorem. As a motivating reminder, the theorem states that for the variational problem

$$u \in X : a(u, v) = L(v) \quad \forall v \in V$$

where X is a Hilbert space, L is bounded and linear, $a : X \times X \to \mathbb{R}$ is bilinear, and (ellipticity) there exists $\alpha > 0$ such that $a(v, v) \geq \alpha ||v||_X^2$ for all $v \in X$, there exists a unique solution of the variational problem.

But in general, the problem is as follows. Given X, Y and $B : X \to Y$, under what conditions is the problem

$$u \in X : Bu = f \in Y$$

uniquely solvable and has continuous dependence on the data, meaning $||u||_X \leq C||f||_Y$. And we want to establish necessary & sufficient conditions for this. This more general problem includes problems from linear algebra as well.

We first rewrite this in a variational form:

$$u \in X : (Bu, v)_Y = (f, v)_Y \quad \forall v \in Y$$

Suppose Y is a normed linear space with scalar product $(., .)_Y$ and similarly for X. Furthermore, the blinear form is $B(u, v) = (Bu, v)_Y$ for all $u \in X, v \in Y$ and the lienar form $L : Y \to \mathbb{R}$ is $L(v) = (f, v)_Y$. We suppose L is linear and continuous

$$|L(v)| \leq \Lambda ||v||_Y \quad \forall v \in Y$$

and B is bilinear and continuous with

$$|B(u, v)| \leq M||u||_X \cdot ||v||_Y \quad \forall u \in X, v \in Y$$

Unlike Lax-Milgram, allow the test space X to be different from the trial space Y. Also, we can't meaningfully talk about ellipticity since $X \neq Y$ and $B(v, v)$ doesn't make sense. So, that condition needs to be replaced by others.

From the original problem, we need 2 conditions. First, that for all $f \in Y$ there exists at least one $u \in X$ such that $Bu = f$ ie

$$\text{Rg}(B) = \{f \in Y : \exists u \in X : Bu = f\} = Y$$

and need unique solutions, so if $u, \tilde{u} \in X$ satisfy $Bu = f = B\tilde{u}$, then $Bu - B\tilde{u} = B(u - \tilde{u}) = 0 \iff u - \tilde{u} = 0$. In other words, $\ker(B) = \{0\}$. This is necessary for unique solvability. These algebraic conditions say nothing about continuous dependence, and we want metric conditions. Also the problem is that these can't really be checked efficiently in practice.

From the continuous dependence side, a necessary condition for continuous dependence is that B is bounded below, ie there exists $\alpha > 0$ such that $||Bu||_Y \geq \alpha ||u||_X$ for all $u \in X$, which is similar to ellipticity. If $X = Y$ this actually is ellipticity.

Choosing $v = Bu$,

$$||Bu||_Y = \frac{B(u, Bu)}{||Bu||_Y} \leq \sup_{v \in Y, v \neq 0} \frac{(Bu, v)_Y}{||v||_Y} \leq \sup_{v \in Y, v \neq 0} \frac{||Bu||_Y ||v||_Y}{||v||_Y} = ||Bu||_Y.$$

So

$$||Bu||_Y \geq \alpha ||u||_X \iff \sup_{v \in Y, v \neq 0} \frac{B(u, v)}{||v||_Y} \geq \alpha ||u||_X \quad \forall u \in X$$

or a more common formulation:

$$\inf_{u \in X, u \neq 0} \sup_{v \in Y, v \neq 0} \frac{B(u, v)}{||u||_X ||v||_Y} \geq \alpha > 0$$

which is the notorious inf-sup condition and is completely equivalent to continuous dependence on given data. In the case $X = Y$, if B is elliptic, then this condition is automatically upheld. It also implies algebraic uniqueness.

However, note that this doesn't say anything about the range condition since a dimension can always be added to Y and the inf-sup condition will always hold.

We claim that the inf-sup condition implies that $\text{Rg}(B)$ is closed. It suffices to show that if $\{f_n\} \subset \text{Rg}(B) : f_n \to f \in Y$, then $f \in \text{Rg}(B)$. Since $f_n \in \text{Rg}(B)$, then there exists $u_n \in X : Bu_n = f_n$, and from the inf-sup condition

$$\alpha||u_n - u_m||_X \leq ||B(u_n - u_m)||_Y \implies ||u_n - u_m||_X \leq \frac{1}{\alpha}||f_n - f_m||_Y \to 0$$

So since $\{f_n\}$ is Cauchy, $\{u_n\}$ is Cauchy, and assuming X is a Hilbert space, completeness gives that there exists $u \in X$ such that $u_n \to u$.

Now consider

$$||Bu - f||_Y \leq ||_Y \leq ||Bu - Bu_n||_Y + ||Bu_n - f||_Y \leq C||u - u_n||_X + ||f_n - f||_Y \to 0$$

Hence there exists $u \in X$ such that $Bu = f$ and so $\text{Rg}(B)$ is closed. It remains to get a checkable condition such that $\text{Rg}(B) = Y$.

Expanding as a direct sum with the orthogonal complement,

$$Y = \text{Rg}(B) \oplus \text{Rg}(B)^\perp$$

so we need a condition that guarantees $\text{Rg}(B)^\perp = \{0\}$. But for any $w \in X$, $B(w,v)_Y = 0$ if $v \in \text{Rg}(B)^\perp$. The Babuska variant of the subsequent theorem states that for every $v \in Y \setminus \{0\} \exists w \in X : B(w,v) \neq 0$ and this leads to the Babuska-Banach-Necas theorem, which here is shown for Hilbert spaces, but holds for Banach spaces as well.

Theorem 0.1. Suppose X, Y are Hilbert spaces such that $B : X \times Y \to \mathbb{R}$ is continuous and bilinear, $L : Y \to \mathbb{R}$ is continuous and linear, there exists $\alpha > 0$ such that

$$\sup_{v \in Y \setminus \{0\}} \frac{B(u,v)}{||v||_Y} \geq \alpha||u||_X \quad \forall u \in X$$

and if $v \neq 0$, then $\sup_{w \in X} B(w,v) > 0$, which is equivalent to $\text{Rg}(B)^\perp = \{0\}$.

With these conditions, there exists a unique $u \in X$ such that

$$B(u,v) = L(v) \quad \forall v \in Y$$

and

$$||u||_X \leq \frac{1}{\alpha}||L||_{y^*}$$

These are necessary and sufficient conditions, plus they give checkable conditions on B.

To finalize the proof, let L be as above, then by the Riesz representation theorem, there exists $f \in Y$ such that $L(v) = (f,v)_Y$ for all $v \in Y$ with $||f||_Y = ||L||_{Y^*}$ From the sup condition, there exists $u \in X : Bu = f$. Since the inf-sup condition implies that $\ker(B) = \{0\}$, u is unique. Finally, by the inf-sup condition, $\alpha||u||_X \leq ||Bu||_Y = ||f||_Y = ||L||_{y^*} \implies ||u||_X \leq \frac{1}{\alpha}||L||_{y^*}$.

Now we do an example which cannot be done by Lax-Milgram. Suppose $-u' = f$ on $I = (0,1)$ with $u(0) = 0$. If f is smooth, then a solution exists, but now suppose $f \in H^{-1}(I)$, meaning it is very unsmooth, it could even be the Dirac delta function. Choose $v \in H^1(I)$, then $(f,v) = (-u',v) = u(1)v(1) + (u,v')$. Require $v(1) = 0$ to get the variational problem $B(u,v) = L(v)$ where $u \in L^2(I)$ and $Y = \{v \in H^1(I) : v(1) = 0\}$, a closed subspace of $H^1(I)$, and hence also a Hilbert space. Here $B(u,v) = (u,v'), L(v) = (f,v)$. The linearity and continuity conditions are easy to check. The inf-sup condition can be checked with: let $u \in X = L^2$ be given and choose

$$v(x) = -\int_x^{-1} u(s)ds$$

Then

$$\frac{B(u,v)}{||v||_Y} = \frac{(u,v')}{||v||_Y} = \frac{||u||^2}{||v||_Y} \geq \frac{||u||}{\sqrt{2}}.$$

Since $||v||_Y^2 = ||v||^2 + ||v'||^2 = ||v||^2 + ||u||^2 \leq 2||u||^2$,

$$|v(x)| \leq \int_x^1 |u(s)|ds \leq (1-x)||u|| \leq ||u|| \implies ||v|| \leq \|u\|.$$

This also suggests the choice $\alpha = 1/\sqrt{2}$.

For the sup condition: let $v \in Y$ be given and non-zero, then choose $w(x) = v'(x)$ so $B(w,v) = (w,v') = ||v'||^2 \geq 0$. Equality holds if and only if $v' = 0 \iff v = 0$ since $v(0) = 0$. But since $v \neq 0$, then $B(w,v) > 0$.

Hence all conditions for BBN are met and there exists a unique solution $u \in L^2$ that depends continuously on the data and $||u|| \leq \sqrt{C}_f$.

Lecture 3 9/21:

We restate BBN. Given a variational problem to find $u \in X$ such that $B(u,v) = L(v)$ for all $v \in Y$ where

- X, Y are Hilbert spaces
- $B : X \times Y \to \mathbb{R}$ is bilinear and continuous
- $L : Y \to \mathbb{R}$ is linear and continuous
- $\sup_{v \in Y \setminus \{0\}} \frac{B(u,v)}{||v||_Y} \geq \beta ||u||_X \quad \forall u \in X, \beta > 0$
- For all $v \in Y \setminus \{0\}$, $\sup_{w \in X} B(w,v) > 0$

These conditions combined imply that there exists a unique u satisfying the variational problem and that the result depends continuously on the given data: $||u||_X \leq 1/\beta ||L||_{Y^*}$.

When $X \neq Y$, the resulting numerical methods are called Petrov methods rather than just Galerkin methods.

We now do an example. Suppose $X = \mathbb{R}^2, Y = \mathbb{R}^3$ and define $B : X \to Y$ by

$$Bx = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

We ask the question of when the problem $Bx = y$ is solvable. Define

$$B(x,y) = y^T Bx; ||Bx||^2 = x_1^2 + x_2^2 + (x_1 + x_2)^2 \geq x_1^2 + x_2^2 = ||x||^2$$

so the $\inf - \sup$ condition holds ie

$$\sup_{y \neq 0} \frac{y^T Bx}{||y||} \geq \frac{(Bx)^T (Bx)}{||Bx||} = ||Bx|| \geq ||x|| \implies \beta = 1.$$

From undergraduate linear algebra, we know that this problem doesn't actually have a unique solution, so we expect the sup condition to fail. In particular, for all $y \in \mathbb{R}^3$, $\sup_{x \in \mathbb{R}^2} y^T Bx > 0$. Indeed,

$$y^T Bx = x_1(y_1 + y_3) + x_2(y_2 + y_3)$$

is equal to 0 if $y_1 + y_3 = y_2 + y_3 = 0$ for all x. This suggests that the problem is not well-posed and it fails BBN, which gives necessary and sufficient conditions for the problem. In particular, $\text{Rg}(B) \neq Y$ and specifically is not solvable whenever $y_1 + y_3 = y_2 + y_3 = 0$. Or even more generally,

$$y \propto \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

then the problem has no solution. However, we can restore the condition that $\text{Rg} = Y$ if we restrict to a subset of Y. Specifically, choosing

$$Y = \left\{ y \in \mathbb{R}^3 : y^T \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0 \right\} = \{y \in \mathbb{R}^3 : y_1 + y_2 = y_3\}$$

gives unique solvability with $B : \mathbb{R}^2 \to Y$.

A more non-trivial example follows: the wave equation

$$\partial_t^2 u - \Delta u = f(x),\ u : [0,T] \times \Omega$$

subject to $u = 0$ on $[0,T] \times \partial\Omega$ and some other initial conditions. A common simplification is the time harmonic case with setting

$$u(x,t) = e^{ikt}u(x),\ k \in \mathbb{R}$$

which gives the problem

$$\begin{cases} -\Delta U - k^2 U = F & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

This is the Helmholtz equation, is there a unique solution? We transfer to the variational form for $u \in X = H_0^1(\Omega)$:

$$\begin{aligned} (F, v) &= (-\Delta U, v) - k^2(U, v) \\ &= (\nabla U, \nabla v) - k^2(U, v) \\ &= B(u, v)\ \forall v \in H_0^1(\Omega) \end{aligned}$$

Note that Lax-Milgram is not applicable because of coercivity problems with the $-k^2$ coefficient.

Let $\varphi \in H_0^1 \setminus \{0\}$. Then $B(\varphi, \varphi) = ||\nabla\varphi||^2 - k^2||\varphi||^2 < 0$ for

$$k^2 \geq \frac{||\nabla\varphi||}{||\varphi||}$$

as an illustration of the fact that this is not an elliptic problem in all cases.

We can't immediately expect unique solvability. In fact, a perturbation by an eigenfunction leads to a loss of uniqueness:

$$\exists \varphi \in H_0^1(\Omega) \setminus \{0\} : B(\varphi, v) = 0 \forall v \in H_0^1(\Omega).$$

then

$$B(\varphi, v) = 0 \iff (\nabla\varphi, \nabla v) = k^2(\varphi, v)$$

and corresponds to the PDE

$$-\Delta\varphi = k^2\varphi$$

with $\varphi = 0$ on $\partial\Omega$. This means that φ is an eigenfunction and $k\varphi$ is an eigensolution of $-\Delta$, meaning no unique solvability for the original problem. But suppose k^2 is not an eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, then does there exist a solution?

Let $v \in H_0^1(\Omega) \setminus \{0\}$ and seek a $w \in H_0^1(\Omega) : B(w, v) > 0$. Suppose this is not possible and that $B(w, v) = 0$ for all $w \in H_0^1(\Omega)$. This happens if and only if $(\nabla w, \nabla v) - k^2(w, v) = 0$ which happens if and only if (v, k^2) are an eigenpair for $-\Delta$, a contradiction. Now we verify the $\inf - \sup$ condition. The contradiction above implies that there exists $\alpha > 0$ such that

$$\sup_{v \in H_0^1 \setminus \{0\}} \frac{B(u, v)}{||v||_{H^1}} \geq \alpha ||u||_{H^1}$$

for all $u \in H_0^1(\Omega)$. Assume for the sake of contradiction that this is not possible. Let $n \in \mathbb{N}$, then there exists $u_n \in H_0^1(\Omega)$ such that

$$\sup_{v \in H_0^1 \setminus \{0\}} \frac{B(u_n, v)}{||v||_{H^1}} < \frac{1}{n} ||u_n||_{H_0^1}$$

Without loss of generality, assume $||u_n||_{H_0^1} = 1$ such that

$$\sup_{v \in H_0^1 \setminus \{0\}} \frac{B(u, v)}{||u||_{H^1}} < \frac{1}{n}$$

By the Rellich-Kondrashov theorem, there exists a subsequence (which we abuse notation for and label $\{u_n\}$ again) such that $u_n \to u_0 \in L^2$. Let $v \in H_0^1$ then

$$\left|\frac{B(u_n, v)}{||v||_{H^1}}\right| < \frac{1}{n}$$

So the sequence

$$\frac{B(u_n, v)}{||v||_{H^1}} \to 0$$

as $n \to \infty$ for all $v \in H_0^1(\Omega)$. Then

$$B(u_n, v) = (\nabla u_n, \nabla v) - k^2(u_n, v) = (u_n, -\Delta v - k^2 v)$$

for all $v \in C_0^\infty$ because we need more regularity and $u_n \to u_0$ while $-\Delta v - k^2 v \in C_0^\infty$. Then applying a distributional derivative ∇u_0,

$$B(u_n, v) = (u_0, \Delta v - k^2 v) = (\nabla u_0, \nabla v) - k^2(u_0, v).$$

Now we have that u_0 satisfies $\nabla u_0, \nabla v) - k^2(u_0, v) = 0$ for all $v \in C_0^\infty$ if and only if $u_0, -k^2$ is an eigensolution for $-\Delta u$, but this contradicts the original assumption, so it follows that $u_0 = 0$.

Now choose $v = u_n$, then

$$\sup_{v \in H_0^1 \setminus \{0\}} \frac{B(u_n, v)}{||v||_{H^1}} < \frac{1}{n} \implies |B(u_n, u_n)| = \left| ||\nabla u_n||^2 - k^2 ||u_n||^2 \right| < \frac{1}{n}$$

from which it follows that $||u_0||^2 = 0$ and therefore $||u_n||_{H^1} \to 0$, which contradicts the assumption that $||u_n||_{H^1} = 1$.

This is an example of a typical compactness-type argument in the literature. Here we summarize the result. Suppose k^2 is not an eigenvalue of $-\Delta$, then for all $f \in H^{-1}(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega)$ such that $(\nabla u, \nabla v) - k^2(u, v) = F(v)$ for all $v \in H_0^1(\Omega)$ and it depends continuously on the data in the sense that $||u||_{H_0^1} \leq C||H||_{H^{-1}}$.

If k^2 is an eigenvalue, the solution is not unique up to addition of the corresponding eigenfunction.

We now consider an extension of these results. Specifically, we consider the adjoint problem for the original variational problem. Namely, to find a $v \in Y$ such that

$$B(u, v) = \tilde{L}(u)$$

for all $u \in X$ where $\tilde{L}$ is bounded. We conjecture that

- $\exists \beta > 0 : \sup_{v \in Y \setminus \{0\}} : \frac{B(u,v)}{||v||_Y} \geq \beta ||u||_X$
- $\sup_{u \in X \setminus \{0\}} B(u, v) > 0 \forall v \in Y \setminus \{0\}$

together imply

$$\sup_{x \in X \setminus \{0\}} \frac{B(u, v)}{||u||_X} \geq \tilde{\beta} ||v||_Y \quad \forall v \in Y$$

We claim that there is automatic well-posedness of the adjoint problem from BBN. To prove this, let $v \in Y \setminus \{0\}$ and define a linear functional $L : Y \to \mathbb{R}$ by $L(w) = (v, w)_Y$ for $w \in Y$ and $||L||_{Y^*} = ||v||_Y$. Well-posedness of the original problem and continuous dependence imply that $||u|_X \leq \frac{1}{\beta} ||L||_{Y^*} \leq \frac{1}{\beta} ||v||_Y$ and $u \neq 0$ because $L \neq 0$ Particularly,

$$\frac{B(u, v)}{||u||_X} = \frac{||v||_Y^2}{||u||_X} \geq \frac{\beta ||v||_Y^2}{||v||_Y} = \beta ||v||_Y$$

So there is no real asymmetry in conditions and the theorem applies equally well to the adjoint problem. In particular,

$$\sup_{u \in X \setminus \{0\}} \frac{B(u, v)}{||u||_X} > \beta ||v||_Y$$

and

$$\exists \beta > 0 : \sup_{v \in Y \setminus \{0\}} \frac{B(u, v)}{||v||_Y} \geq \beta ||u||_X > 0$$

are the same result.

Lecture 4 9/28:

The variational form leads us to question the convenience of using BBN. We do several derivations for the Stokes equations, Darcy flow, and the reduced Maxwell equation for magnetostatics to show that all these problems reduce to the mixed form

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v)_V \\ b(u,q) &= (g,q)_M \end{aligned}$$

for all $(v,q) \in X = V \times M$. We now want to find simpler conditions to check for well-posedness of this problem.

The Brezzi theory determines when the mixed form is well-posed but is really a rehash of BBN. In the above mixed form, we collapse to a single problem with

$$B((u,p),(v,q)) = a(u,v) + b(v,p) + b(u,q), \quad L(v,q) = (f,v)_V + (g,q)_M$$

where B and L are continuous and (bi)linear if and only if $a(\cdot,\cdot), b(\cdot,\cdot)$ are bilinear and continuous. Also by Cauchy-Schwarz,

$$|L(v,q)| \leq ||f||_V \cdot ||v||_V + ||g||_M \cdot ||q||_M \leq C(f,q) \cdot ||(v,q)||_X.$$

The sup condition is equivalent to

$$0 < \sup_{(u,p)\in X} B(u,p,0,q) = \sup_{u\in V} B(u,q)$$

Choosing $u = 0$ in the inf-sup condition, we have the Babuska-Brezzi condition

$$\beta ||p||_M \leq \sup_{v\in V\setminus\{0\}} \frac{b(v,p)}{||v||_V}$$

Originally, this was posed with V-ellipticity on $a(\cdot,\cdot)$ as well. But, to reduce the assumptions on that form, consider the subspace

$$V_0 = \{v \in V : b(v,q) = 0 \forall q \in M\} \subset V$$

so V_0 is the kernel of the bilinear form $v \mapsto b(v,q)$. This suggests the assumption $a(v,v) \geq \alpha ||v||_V^2$ for all $v \in V_0$. So a should be elliptic on the kernel of b. Also, V_0 is a closed subspace since b is continuous and we can decompose $V = V_0 \oplus V_0^{\perp}$. Letting $v = v_0 + \tilde{v}$, we have

$$||v||_V^2 = ||v_0||_V^2 + ||\tilde{v}||_V^2$$

so $b(v,q) = b(\tilde{v},q)$. From which BBN simplies to

$$\begin{aligned} \beta ||p||_M &\leq \sup_{v\in V\setminus\{0\}} \frac{b(v,p)}{||v||_V} \\ &= \sup_{v_0+\tilde{v}\neq 0} \frac{b(v,q)}{\sqrt{||\tilde{v}||_V^2}} \\ &= \sup_{\tilde{v}\in \tilde{V}\setminus\{0\}} \frac{b(\tilde{v},q)}{||\tilde{v}||_V} \end{aligned}$$

so to conclude, we have these conditions:

$$\begin{aligned} &\sup_{\tilde{u}\in\tilde{V}} b(\tilde{u},q) > 0 \quad \forall q \in M \setminus \{0\} \\ &\tilde{v} \in \tilde{V} : b(\tilde{v},q) = l(q) \quad \forall q \in M \end{aligned}$$

which give conditions on the problem $q \in M : b(\tilde{w},q) = l(\tilde{w})$ is solvable by standard BBN.

Lecture 5 8/5:

An additional comment to be made on mixed-order/saddle-point form problems is that the second equation can be considered to be a constrained. In essence, the mixed form can be thought of as quadratic functional minimization against a linear constraint.

After a similar reconsideration of the BBN conditions and the mixed-form decomposition, we turn to stating Brezzi's 1974 theorem as follows. Suppose $a : V \times V \to \mathbb{R}, b : V \times M \to \mathbb{R}$ are continuous and bilinear where V and M are Hilbert. Suppose there exists $\beta > 0$ such that

$$\sup_{v \in V \setminus \{0\}} \frac{b(v,q)}{||v||_V} \geq \beta ||q||_M \quad \forall q \in M$$

and there exists an $\alpha > 0$ such that

$$\sup_{v_0 \in V_0 \setminus \{0\}} \frac{a(u_0, v_0)}{||v_0||_V} \geq \alpha ||u_0|| V$$

where

$$V_0 = \{v \in V : b(v,q) = 0 \quad \forall q \in M\}$$

which in other words means that a is elliptic on the kernel of b. Finally, suppose

$$\sup_{u_0 \in V_0} a(u_0, v_0) > 0 \quad \forall v_0 \in V_0 \setminus \{0\}$$

then there exists a unique solution $(u,p) \in V \times M$ to the problem

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v)_V \\ b(u,q) &= (g,q)_M \end{aligned}$$

for all $(v,q) \in V \times M$ and it depends continuously on $||f||_v + ||g||_M$.

To prove this, we need only check the inf-sup condition for B. Let $(u,p) \in V \times M$ be given. Construct (v,q) and observe that

$$a(u_0,v_0) + a(u_0,\hat{v}) + a(\hat{u},v_0) + a(\hat{u},\hat{v}) + b(\hat{u},q) + b(\hat{v},p) \geq ||u_0||_V^2 + ||\hat{u}||_V^2 + ||p||_M^2.$$

By inf-sup there exists a $\hat{v} \in \hat{V}$ such that $b(\hat{v},q) = (p,q)_m$ for all $q \in M$. Hence $b(\hat{v},p) = ||p||_M^2$. Now since $a : V_0 \times V_0 \to \mathbb{R}$ satisfies BBN, there exists a $v_0 \in V_0$ such that

$$a(w_0,v_0) + a(w_0,\hat{v}) = (u_0,w_0) \quad \forall w_0 \in V_0$$

with

$$||v_0||_V \leq \frac{1}{\beta}[||u_0||_V + C_a/\beta ||p||_M]$$

Using the inf-sup again, there exists a $q \in M$ such that

$$a(\hat{w},v_0) + a(\hat{w},\hat{v}) + b(\hat{w},q) = (\hat{u},\hat{w})_V \quad \forall \hat{w} \in \hat{V}$$

with

$$\begin{aligned} ||q||_M &\leq \frac{1}{\beta}[||\hat{u}||_V + C_a||v_0||_V + C_a||\hat{v}||_V] \\ &\leq C(\beta, C_a)[||u_0||_V + ||\hat{u}||_V + ||p||_M] \\ &\leq \sqrt{2}C(\beta, C_a)||(u,p)||_{V \times M} \end{aligned}$$

which implies

$$B(u,p,v,q) \geq ||u_0||_V^2 + ||\hat{u}||_V^2 + ||p||_M^2 \geq ||(u,p)||_{V \times M} \cdot \frac{1}{C}||(v,q)||_{V \times M}$$

from which it follows that

$$\sup_{(v,q) \in V \times M} \frac{B(u,p,v,q)}{||(v,q)||_{V \times M}} \geq \frac{1}{C}||(u,p)||_{V \times M}$$

and the sup condition is similar.

Lecture 6 10/12:

We now discuss the Galerkin approximation of variational problems, not necessarily being constrained to the elliptic case. As an example, consider

$$\begin{cases} -u' &= f \quad \text{on } I = (0,1) \\ u(0) &= 0 \end{cases}$$

which admits two variational forms:

- (1) $u \in X_1 = L_2(I) : (u, v') = (f, v)$ for all $v \in Y_1 = \{v \in H^1(I) : v(1) = 0\}$
- (2) $u \in X_2 = \{w \in H^1(I) : w(0) = 0\} : (-u', v) = (f, v)$ for all $v \in Y_2 = L_2(I)$.

The two bilinear forms are adjoints to each other. In the following, we work with the variational formulation (1). Construct a Galekin subspace $X_1^h \subset X_1$ and subdivide I into uniform elements to get $Y_1^h \subset Y_1$ where the subdivision is of width $h > 0$.

Choose

$$X_1^h = \{w_h \in X_1 : w_h \text{ is continuous and piecewise linear}\}$$

and

$$Y_1^h = \{v_h \in Y_1 : \text{ is continuous and piecewise linear}, v_h(1) = 0\}$$

The Galerkin scheme is to solve

$$u_h \in X_1^h : (u_h, v') = (f, v) \quad \forall v \in Y_1^h$$

One key observation to make here is that this problem is not self-adjoint. The above leads to a rectangular matrix system because

$$\dim X_1^h = \dim Y_1^h + 1$$

and thus we do not expect for this problem to have a unique solution. Concretely, defining $w_h \in X_1^h$ as a sawtooth oscillating function, $(w_h, v_h') = 0$ has a vanishing average on each element. However, there are no other linearly independent alternatives, so adding this up to a constant gives non-unique solutions. And since there is non-uniqueness of solutions, an implementation of this will not have convergence numerically either. This is due ot a failure of the inf-sup condition, ie

$$\sup \frac{B_1(w_h)}{||v_h'||} = 0.$$

Using piecewise linears again for (2),

$$X_2^h = \{v \in X_2 : v \text{ pw linear}\}$$
$$Y_2^h = \{v \in Y_2 : v \text{ pw linear}\}$$

where $\dim X_2^h = \dim Y_2^h - 1$. There is no existence of a solution due to dimension mismatch here. This is because of a failure on the sup condition, ie range$(B) \neq Y$.

This leads to the consideration of the trade off of choosing spaces in such a way that they respect BBN and specifically the sup and inf-sup conditions. Making X_h smaller increases the stability of the numerical method but also makes the range smaller and at a certain point, u_h might not just exist at all. On the other hand, making Y_h bigger increases the constant β_h in the inf-sup condition, which improves stability,

So suppose we try

$$X_1^h = \{x \in X_1 : \text{ discontinuous pw constants } \}$$
$$Y_1^h = \{v \in Y_1 : \text{ cts piecewise linears } \}$$

then recall $u \in L^2$, we seek v such that

$$\frac{B(u,v)}{||v||_Y} \geq \beta ||u||_X$$

and

$$B(u,v) = (u, v') \geq \beta ||u|| \cdot ||v'||$$

so in particular, $v_h' = u_h$, which implies the choice $v(x) = -\int_x^1 u(s)ds$. We would like to discretize this problem, but there's no clear way to discretize L^2. However, note that this scheme is well posed because $\dim X_1^h$ is the number of elements and $\dim Y_1^h$ is 1 minus the number of nodes because of the constraint $Y(1) = 0$ and hence is equal to $\dim X_1^h$.

We call a scheme uniformly stable in h if and only if there exists $\beta > 0$ such that $\beta_h \geq \beta$ for all $h > 0$. In this particular case, from the inf-sup condition, letting $u_h \in X_h^1$ be given and choose $v_h \in Y_1^h$ then $v_h(x) = -\int_x^1 u_h(s)ds$, then

$$\frac{B(u_h, v_h)}{||v_h||_Y} = \frac{||u_h||^2}{||v_h'||} = ||u_h|| \implies \beta_h = 1 \quad \forall h$$

so we have shown that this scheme is uniformly stable. We can also show that this scheme is well-posed from the sup condition. Letting $v_h \in Y_1^h$ be non-zero, choose $u_h = v_h' \in X_1^h$, then

$$B(u_h, v_h) = ||v_h'||^2 > 0.$$

For accuracy and convergence, we need a generalized Cea's lemma that will give

$$||u - u_h|| \leq ||u - v_h|| \quad \forall v_h \in X_h$$

for symmetric elliptic problems, which motivates the Babuska-Aziz approximation.

Consider $u \in X : B(u, v) = L(v)$ for all $v \in Y$ where B is bilinear and continuous, L is linear and continuous and assume u exists and is unique. Now let $X_h \subset X$ and $Y_h \subset Y$ be finite dimensional subspaces such that

- $\exists \beta_h > 0$ such that
$$\sup_{v_h \in Y_h \setminus \{0\}} \frac{B(u_h, v_h)}{||v_h||_Y} \geq \beta_h ||u_h|| \quad \forall u_h \in X_h$$
- for all $v_h \in Y_h \setminus \{0\}$, $\sup_{w_h \in X_h} B(w_h, v_h) > 0$

Then there exists a unique $u_h \in X_h$ such that $B(u_h, v_h) = L(v_h)$ for all $v_h \in Y_h$ and

$$||u - u_h||_X \leq \left(1 + \frac{C_B}{\beta}\right) ||u - \tilde{u}_h||_X$$

for all $\tilde{u}_h \in X_h$.

Since $X_h \subset X, Y_h \subset Y$, B and L are linear and continuous. Thus there exists $C_\beta > 0$ such that

$$|B(u, v)| \leq C_\beta ||u||_X \cdot ||v||_Y \quad \forall u \in X_h, v \in Y_h$$

From the given conditions, BBn says u_h exists and is unique. Let $\tilde{u}_h \in X_h$. Then

$$||u_h - \tilde{u}_h||_X \leq \frac{1}{\beta_h} \sup_{v_h \in Y_h} \frac{B(u_h - \tilde{u}_h, v_h)}{||v||_Y}$$

Then taking $B(u_h - \tilde{u}_h, v_h) = B(u - \tilde{u}_h, v_h)$, we have

$$||u_h - \tilde{u}_h||_X \leq \frac{C_\beta}{\beta_h} ||u - \tilde{u}_h||_X$$

and by triangle inequality

$$||u - u_h||_X \leq ||u - \tilde{u}_h||_X + ||\tilde{u}_h - u_h||_X \leq \left(1 + \frac{C_\beta}{\beta_h}\right) ||u - \tilde{u}_h||_X$$

As the model problem, let $B(u, v) = (u, v')$ where X_h are piecewise constants and Y_H are piecewise linears. We showed that C_β is a constant and $\beta_h = 1$. It remains to estimate $||u - \tilde{u}_h||_{L^2}$ where $\tilde{u}_h$ is a piecewise constant. We take averages to approximate constants. Choose

$$\tilde{u}_h : \tilde{u}_h|_{I_j} = \frac{1}{h} \int_{I_J} u(s)ds$$

then by Poincare $||u - \tilde{u}_h||_{I_j} \leq Ch_j||u'||_{I_j}$ and so

$$\begin{aligned} ||u - \tilde{u}_h|^2_{L^2} &= \sum_{I_j} ||u - \tilde{u}_h||^2_{I_j} \\ &\leq C \sum_{I_j} h_j^2 ||u'||^2_{I_j} \\ &\leq Ch^2 \sum_{h_j} ||u'||^2_{I_j} \\ &= Ch^2 ||u'||^2 \end{aligned}$$

so $||u - \tilde{u}_h||_{L^2} \leq Ch||u'||$ provided $u \in H^1$. Here, C depends on C_p, C_β, C_h.

Another example is an extension of this to saddle-point problems Consider the problem of finding $(u, p) \in V \times M$ such that

$$\begin{aligned} &a(u, v) + b(v, p) = (f, v) \quad \forall v \in V \\ &b(u, q) = (g, q) \quad \forall q \in M \end{aligned}$$

and assume conditions of well-posedness. Choose $V_h \subset V$ and $M_h \subset M$ and apprximate using a Galekin scheme, which amounts to solving the problem

$$\begin{aligned} &a(u_h, v_h) + b(v_h, p_h) = (f, v_h) \quad \forall v_h \in V_h \\ &b(u_h, q_h) = (g, q_h) \quad \forall q_h \in M_h \end{aligned}$$

We are curious about finding the conditions under which this scheme is well-posed. Define the discrete kernel $V_0^h = \{v \in V_h : b(v, q) = 0 \forall q \in M_h\}$. Note that V_0^h is not a subspace of V_0 in general! It's simple to check the Brezzi conditions and collapsing to a big bilinear form, that BBN conditions are met. We have that the Galerkin approximation of the saddle point problem admits a unique approximation $(u_h, p_h) \in V_h \times M_h$ and

$$||(u, p) - (u_h, p_h)||_{V \times M} \leq C(\beta_h, C_\beta, C_a, \cdots)||(u, p) - (\tilde{u}_h, \tilde{p}_h)||_{V \times M}$$

for all $(\tilde{u}_h, \tilde{p}_h) \in V_h \times M_h$.

As another example, we do Stokes flow on $\mathbb{R}^1$. Assume $u = \begin{bmatrix} u(x) \\ 0 \end{bmatrix}$ is univariate flow that is governed by

$$\begin{cases} -\varepsilon u'' + p' &= f \text{ on } I = (0, 1) \\ u' &= g \end{cases}$$

subject to the initial values $u(0) = u(1) = 0$. The corresponding saddle point formulation is to find $(u, p) \in H_0^1 \times L^2/\mathbb{R}$ where the quotient is to enforce an average value of 0.

Choose V_h to consists of continuous piece wise linears from H_0^1 and M_h as piecewise constants from $L^2/\mathbb{R}$ in order to satisfy the inf-sup condition on $b(v, p) = (v', p)$. In a similar way to the previous example, $\beta_h = 1$. Then notices that $V_0^h = \{v_h \in V_h : b(v_h, q_h) = 0 \quad \forall q_h \in M_h\} = \{0\}$, so we get trivial ellipticity on the kernel.

Thus the scheme is well-posed and uniformly stable. We also have that

$$||(u, p) - (u_h, p_h)||_{V \times M} \leq C||(u, p) - (\tilde{u}_h, \tilde{p}_h)||_{V \times M}$$

if and only if

$$||u - u_h||_{H^1} + ||p - p_h||_{L^2} \leq C\{||u - \tilde{u}_h||_{H^1} + ||p - \tilde{p}_h||_{L^2}\}$$

which leads to the concept of pressure-robust schemes where pressure accuracies do not affect velocity accuracies. Now choose $\tilde{u}_h$ that are piecewise linear interpolants of u at nodes and $\tilde{p}_h$ are piecewise constant averages of p on elements. Then

$$\int_I \tilde{p}_h = \sum_{I_j} \int_{I_j} \tilde{p}_h = \sum_{I_j} \int_{I_j} p = \int_I p = 0$$

for all $p \in L^2/\mathbb{R}$. Thus $\tilde{p}_h \in M$. We have already shown that $||p - \tilde{p}_h|| \leq Ch||p||_{H^1}$ and $||u - \tilde{u}_h||_{H^1} \leq Ch||u||_{H^2}$. Hence

$$||u - u_h||_{H^1} + ||p - p_h||_{L^2} \leq Ch\{||u||_{H^2} + ||p||_{H^1}\}.$$

Lecture 7 10/19:

We now consider mixed finite element methods for second order elliptic equations in general. But here we limit ourselves to $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ and $f \in L^2(\Omega)$.

The primal mixed form goes as follows. Let $\sigma = \operatorname{grad} u$ so $-\text{div}\sigma = f$ in Ω, then for $(\tau, v) \in L^2(\Omega) \times H_0^1(\Omega)$,

$$\begin{aligned} (\sigma,\tau) - (\tau, \operatorname{grad} u) &= 0 \\ -(\sigma, \operatorname{grad} v) &= -(f, v) \end{aligned}$$

Here, the term "mixed" is used in the sense that there are mixed unknowns. Let $a(\tau,\sigma) = (\tau,\sigma), b(\tau, v) = -(\tau, \operatorname{grad} v)$ and $V = L^2(\Omega)^d \times M$ with $M = H_0^1(\Omega)$. It can be shown that the Brezzi conditions are satisfied with constant $\beta = 1$ and so this problem is well-posed. We discretize this and for a partition P_h, choosing the subspaces

$$\begin{cases} V_h &= \{\tau \in L^2(\Omega)^d : \tau|_k \in P_h(k)^d\} \\ M_h &= \{v \in H_0^1(\Omega) : v|_k \in P_{h+1}(k)\} \end{cases}$$

for every element $k \in P_h$. Check that the discrete inf-sup condition holds with $\beta_h = 1$. By Babuska-Aziz,

$$||u - u_h||_{H_0^1} + ||\sigma - \sigma_h||_{L^2(\Omega)^d} \le Ch^n ||u||_{H^{n+1}(\Omega)}, \quad u \in H^{n+1}(\Omega).$$

This scheme is apparently unfavorable in applications, as will be seen in the homework. Instead, we consider a scheme that is used in practice – the Dual mixed form.

Let $u = \operatorname{grad} u$, so $-\text{div}\sigma = f$ in Ω. Now let $\tau \in L^2(\Omega)^d$ and thus by integration by parts,

$$0 = (\sigma, \tau) + (u, \text{div}\tau)$$

provided $\text{div}\tau \in L^2(\Omega)$. Furthermore, for $v \in L^2(\Omega)$, $-(f, v) = (\text{div}\sigma, v)$. This suggests the alternative mixed scheme to find $(\sigma, u) \in V \times M$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau, u) &= 0 \\ b(\sigma, v) &= -(f, v) \end{aligned}$$

where $a(\sigma,\tau) = (\sigma,\tau), b(\tau, v) = (\text{div}\tau, v)$. This formulation is much more popular than the primal form before because the primary variable of interest is the flux σ, rather than u and the physics is based on a conservation law, here $-\text{div}\sigma = f$ in Ω. As a reminder, given a domain ω the rate of production of material is $\int_\omega f dx$ and the rate of flow of material across the boundary $\partial\omega$ is $\int_{\partial\omega} n \cdot \sigma ds$. Then conservation of mass is given by

$$\begin{aligned} 0 &= \int_\omega f dx + \int_{\partial\omega} u \cdot \sigma ds \\ &= \int_\omega (f + \text{div}\sigma) dx \quad \forall \omega \in \Omega. \end{aligned}$$

This implies $-\text{div}\sigma = f$ in Ω. Thus the dual mixed form has this physics included inherently.

Choose $\omega \in \Omega$ and let

$$v = \begin{cases} 1 & \omega \\ 0 & \Omega \setminus \omega \end{cases} \in M$$

Then from the formulation, $b(\sigma, v) = -(f, v)$ and

$$\int_{\partial\omega} n \cdot \sigma = \int_\omega \text{div}\sigma = -\int_\omega f$$

so the dual mixed form embodies mass conservation, as claimed.

When constructing the Galerkin approximation, $M_h \subset M : M_h = \{v \in M : v|_k \in P_h(k)\}$,

$$b(\sigma_h, v_h) = -(f, v_h) \quad \forall v_h \in M_h,$$

set

$$v_h = \begin{cases} 1 & k \in P_h \\ 0 & \Omega \setminus k \end{cases}$$

which implies mass conservation on each element. But how do we choose V for dual mixed schemes? If we choose $V = H^1(\Omega)^d$ (which implies using continuous piecewise linears), then $\tau \in Y \implies \tau \in L^2(\Omega)^d$ and div$\tau \in L^2(\Omega)$ which implies a, b are continuous and bilinear. For inf-sup, let $v \in L^2(\Omega)$ be non-zero. We want $b(\tau, v) = (\text{div}\tau, v) \geq \beta||v|| \cdot ||\tau||_{H^1(\Omega)^d}$. We want div$\sigma = v$ and

$$||\tau||_{H^1(\Omega)^d} \leq C||v||_{L^2} \implies (\text{div}\tau, v) = ||v||^2 \geq ||v||\frac{1}{c}||\tau||_{H^1}.$$

We can use the Helmholtz decomposition $\tau = -\operatorname{grad}\varphi, = \operatorname{div}\operatorname{grad}\varphi = v$ which gives a Poisson problem over Ω with $\Delta\varphi = v$ and $\varphi = 0$ on $\partial\Omega$. The regular results for Poisson problems imply that if Ω is convex, then $\varphi \in H^2$ (if not, then Ω can be extended to a convex domain and setting $\varphi = 0$ outside) and also $||\varphi||_{H^2} \leq C||v||_{L^2}$ (this follows from elliptic regularity). Next choose $\tau = -\operatorname{grad}\varphi \in H^1(\Omega)^d$ with

$$||\tau||_{H^1(\Omega)^d} = ||\operatorname{grad}\varphi||_{H^1(\Omega)^d} \leq C||\varphi||_{H^2(\Omega)} \leq \tilde{C}||v||_{L^2(\Omega)}$$

and div$\tau = v$ in Ω. Hence $b(\tau, v) = (\text{div}\tau, v) = ||v||^2_{L^2} \geq ||v||_{L^2}\frac{1}{c}||\tau||_{H^1(\Omega)^d}$. This implies that

$$\sup_{0\neq\tau\in H^1(\Omega)^d} \frac{b(\tau, v)}{||\tau||_{H^1(\Omega)^d}} \geq \frac{1}{\tilde{C}}||v||_{L^2}$$

so the inf-sup condition holds. Also $a(\sigma, \tau) = (\sigma, \tau)$ is positive definite.

However, consider the Poisson problem on the three-quarter circle with 0 boundary conditions and $f \in L^2(\Omega)$. The solution is $u(r, \theta) = (r^{2/3} - r^{5/3}) \sin 2/3\theta$. Then $\sigma = \operatorname{grad} u \sim r^{-1/3} \in L^2(\Omega)^d$ but $\sigma \notin H^1(\Omega)^d$. The problem with this counter example is that the ellipticity wasn't checked on the kernel. Specifically,

$$V_0 = \{\tau \in H^1(\Omega)^d : (\text{div}\tau, v) = 0\} = \operatorname{curl} H^2$$

in two dimensions.

The general problem here is that the norm on H^1 is too strong and so we cannot choose something from $H^1(\Omega)^d$, but using $L^2(\Omega)^d$ leads to a norm that is too weak because $b(\cdot, \cdot)$ is no longer continuous... To remedy this situation, choose

$$V = H(\text{div}) = \{\tau \in L^2(\Omega)^d : \text{div}\tau \in L^2(\Omega)$$

where the norm is defined by

$$||\tau||^2_{H(\text{div})} = ||\tau||^2_{L^2} + ||\text{div}\tau||^2_{L^2}.$$

In this space, b is continuous and bilinear, satisfies the inf-sup condition as

$$||\tau||_{H(\text{div})} \leq ||\tau||_{H^1} \leq C||v||_{L^2}$$

where the proof is identical to before. It also satisfies ellipticity on the kernel - there exists $\alpha > 0$ such that $a(\tau, \tau) \geq \alpha||\tau||^2_{H(\text{div})}$ for all $\tau \in V_0$ where

$$V_0 = \{\tau \in H(\text{div}) : (\text{div}\tau, v) = 0 \forall v \in L^2\} = \{\tau \in H(\text{div}) : \text{div}\tau = 0\}$$

so $||\tau||_{H(\text{div})} = ||\tau||_{L^2}$ for all $\tau \in V_0$ which implies $\alpha = 1$, so there is no degeneration under a at all!

Now the question becomes how to discretize $H(\text{div}, \Omega)$. We show a preliminary lemma. Let $\tau \in H(\text{div}_)$, $\Gamma \subset \Omega$ be any subsurface/submanifold. Then $[n \cdot \tau] = 0$ on Γ, which intuitively means that the normal components from one side of the boundary cancel out with the normal components on the other side of the boundary. To show this, let

$B \subset \Omega$ be any ball, $v \in H_0^1(B)$ and suppose Γ separates B into B_-, B_+. Let $\tau_+ = \tau|_{B_+}, \tau_- = \tau|_{B_-}$. Then

$$\begin{aligned}\int_{\Gamma\cap B_+} v\hat{n}_+ \cdot \tau_+ ds &= \int_{\partial B_+} v\hat{n}_+ \cdot \tau_+ ds \\ &= \int_{B_+} \nabla \cdot (v\tau_+) dx \\ &= \int_{B_+} v\nabla \cdot \tau_+ + \operatorname{grad} v \cdot \tau_+ \\ &\leq ||v||_{L^2(B_+)} \cdot ||\nabla \cdot \tau_+||_{L^2} + ||\nabla v||_{B_+} \cdot ||\tau_+||_{B_+} \\ &\leq ||v||_{H^1(B_+)} \cdot ||\tau_+||_{H(\text{div}, B_+)}\end{aligned}$$

by using Cauchy-Schwarz twice. Since we are assuming $\tau \in H(\text{div}, B)$, the above quantity is finite. Similarly, $\int_{\Gamma\cap B_-} vn_- \cdot \tau_- ds$ is well-defined. Hence,

$$\begin{aligned}\int_{\Gamma\cap B} v \cdot [n \cdot \tau] ds &= \int_{B_+} \nabla \cdot (v\tau_+) + \int_{B_-} \nabla \cdot (v\tau_-) \\ &= \int_B \nabla \cdot (v\tau) = \int_{\partial B} (n \cdot \tau) v = 0\end{aligned}$$

because $v \in H_0^1(B)$.

A key remark is to think of $[n \cdot \tau]$ in terms of the integral above rather than a pointwise evaluation. Also this jump condition is related to mass conservation $\int_{\partial\omega} n \cdot \sigma = 0$ by thinking about the normal components of flow in and out of any boundary being equivalent, eg mass conservation. Specifically,

$$\int_{\partial\omega} n \cdot \tau = n_x \tau_x + n_y \tau_y.$$

Lecture 8 10/26:

We move towards discretization schemes of $H(\text{div})$. Recall last time we showed the following lemma.

Let $\tau \in H(\text{div})$ and $\Gamma \subset \Omega$ be any surface, then

$$[|n \cdot \tau|] = 0$$

on Γ. This means that the tangential components can nbe discontinuous but the normal components are enforced to be continuous. In particular, one can't consider trace operators in this case, but normal components of the trace can be considered successfully. In proving this, we showed that

$$\int_\Omega vn \cdot \tau ds$$

is well-defined for all $v \in H_0^1(B)$. In particular, the above gives the exact sense in which $n \cdot \tau$ is well-defined (in a weak sense, since point-wise evaluations are not possible).

When it comes to choosing the spaces from which functions are used in the approximation, note that both components (normal and tangential) of H^1 functions are continuous and in particular, that an interpolation via functions in H^1 does not lend itself well to interpolating functions with potentially discontinuous tangential components.

Another note is that $n \cdot \tau$ must be continuous between neighboring elements. In particular, that $n\tau$ must be continuous across common faces or edges. Moreover, a discretization of $H(\text{div})$ only requires continuity on edges, but not on vertices of a mesh, which implies tangential component continuity. This suggests that the degrees of freedom should be as

$$\tau \mapsto \int_\gamma n \cdot \tau ds \quad \forall \gamma \subset \partial K$$

ie the 0th moments of $n \cdot \tau$ on edges, for sufficiently smooth τ, in a sense that will be shown later.

Recall that unisolvence requires that the number of degrees of freedom is equal to the degree of the interpolating space. We are expecting vector-valued functions to be performing the interpolation since $H(\text{div})$ consists of vector fields. Using triangles in the mesh, since each element has 3 degrees of freedom, the interpolating space must be of dimension 3, but note that $\mathbb{P}_0 \times \mathbb{P}_0$ has dimension 2 while $\mathbb{P}_1 \times \mathbb{P}_1$ has degree 6. For a tetrahedral mesh, which has 4 degrees of freedom, we have an even worse problem since $\mathbb{P}_0 \times \mathbb{P}_0 \times \mathbb{P}_0$ has dimension 3 while $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ has dimension 64. What's more, the degree for spaces must also be symmetric across components. So, to summarize, the interpolating space must satisfy the following criteria:

- it is vector-valued
- the dimension is $d+1$ over $\mathbb{R}^d$
- it is unisolvent with respect to
$$\tau \mapsto \int_\gamma n \cdot \tau ds \quad \forall \gamma \subset \partial K$$
- and it must approximate $H(\text{div})$

Also as a sidenote, the degree of the moment corresponds to the degree of approximability of the chosen space. The space that solves these problems is given by the 0 Raviart-Thomas space:

$$\mathbb{RT}_0 = \mathbb{P}_0^d \oplus x\mathbb{P}_0$$

The k-th space is given by $\mathbb{P}_k^d \oplus x\mathbb{P}_k$ with the corresponding multiplier space given by the divergence of this, ie $\mathbb{P}_k$. Now we check the conditions of the space.

For unisolvence, can we construct a basis for $\mathbb{RT}_0$ which is Lagrange with respect to degrees of freedom? ie

$$\{\varphi_\gamma : \gamma \text{ is an edge/face}\} : \varphi_\gamma \mapsto \int_{\gamma'} n \cdot \varphi_\gamma ds = \begin{cases} 1 & \text{if } \gamma = \gamma' \\ 0 & \text{else} \end{cases}$$

The answer is yes, with the choice

$$\varphi_\gamma = \frac{1}{d|K|}(x - x_\gamma)$$

where d is the dimension and $|K|$ is the measure of a particular element K. Note that

$$n_\gamma \cdot \varphi_\gamma = \begin{cases} 1 & \text{if } \gamma \neq \gamma' \\ 0 & \text{else} \end{cases}$$

because φ_γ is tangential to γ'. This can also be thought of as the projection of x_γ onto γ, which has a constant normal component. In other words, $n_\gamma \cdot \varphi_\gamma = \text{dist}(x_\gamma, \gamma)$, which is constant on γ, and so

$$\int_\gamma n_\gamma \cdot \varphi_\gamma ds = \frac{|\gamma|}{d|k|}\text{dist}(x_\gamma, \gamma) = 1$$

We now give a more general definition of the Raviart-Thomas element of degree 0. It is given by $\mathbb{RT}_0 = (K, \Sigma, P)$ where K is a simplex in d dimensions,

$$\Sigma = \left\{ \tau \mapsto \int_\gamma n \cdot \tau ds \quad \forall \text{ subsimplices of dimension } d-1 \right\}$$

and $P = \mathbb{P}_0^d \oplus x\mathbb{P}_0$ which has the basis given by

$$\varphi_\gamma = \frac{1}{d|k|}(x - x_\gamma)$$

which are linearly independent. We claim this triple forms a finite element. From the above discussion, it remains to show unisolvence.

Let $\sigma \in P$ and write $\sigma = \sum c_\gamma \varphi_\gamma$ then

$$\int_\gamma n \cdot \sigma ds = 0 \iff c_\gamma = 0$$

Now does this generate an $H(\text{div})$-conforming space on a mesh? Ie do we get $[|n\cdot\sigma|]=0$ across faces? Orienting all edges in code for interface between elements, otherwise there is direction disagreement on interfaces, meaning if two elements are adjacent to one another, one should get a value of c_γ while the other gets $-c_\gamma$. Changing the sign of a negatively-oriented edge corresponds to changing the sign of the basis function. This means the global basis functions are obtained by flipping signs on elements where γ is negatively-oriented.

Let's go back to the dual mixed form. We seek $(\sigma_h, u_h)\in V_h^{RT}\times M_h$ such that

$$(\sigma_h,\tau_h)+(\text{div}\tau_h,u_h)=0$$
$$(\text{div}\sigma_h,v_h)=-(f,v_h)$$

for all $(\tau_h, v_h)\in V_h^{RT}\times M_h$ where

$$V_h^{RT}=\{\varphi_\gamma^g:\gamma\in\text{edges of }T_h\}$$
$$M_h=\{v_h\in L^2(\Omega):v_h|_k\in P_0(k)\forall K\in T_h\}$$

where the latter is chosen to satisfy Brezzi's theorem. Does this scheme satisfy the inf-sup condition with $\beta_h>\beta>0$ ie is it stable? And can we quantify the accuracy of the resulting approximation?

Let $\sigma\in H(\text{div},\Omega)$ be sufficiently smooth such that

$$\sigma\mapsto\int_\gamma n\cdot\sigma ds$$

for all γ which are edges and faces, is well-defined. Define the Raviart-Thomas interpolants

$$\Pi_{RT}\sigma\in\mathbb{RT}_0:\int_\gamma n\cdot\Pi_{RT}\sigma ds=\int_\gamma n\cdot\sigma\forall\gamma$$

since we have a basis,

$$\Pi_{RT}=\sum_\gamma c_\gamma\varphi_\gamma^G$$

where

$$c_\gamma=\int_\gamma n\cdot\sigma ds.$$

Alternatively, we can restrict to a single element k and define an element-level interpolant:

$$\Pi_k^{RT}\sigma(x)=\sum_{\gamma\subset\partial K}C_\gamma\varphi_\gamma(x)$$

where

$$c_\gamma=\int_\gamma n\cdot\sigma,\varphi_\gamma=\frac{1}{d|k|}(x-x_\gamma).$$

We now prove some useful properties of Raviart-Thomas elements.

- For $\gamma'\subset\partial K$,
$$\int_{\gamma'}n\cdot\Pi_{RT}\sigma=\int_{\gamma'}n\cdot\sigma$$
and hence $\Pi_{RT}\circ\Pi_{RT}=\Pi_{RT}$, in other words, it is a projection.
- $\text{div}\Pi_{RT}\sigma|_K=\sum_{\gamma\subset\partial_K}\text{div}\varphi_\gamma|_K\int_\gamma n\cdot\sigma ds$. In particular, the divergence of the Raviart-Thomas basis functions is independent of x and is the same for all γ. This can be expanded further:
$$\begin{aligned}\text{div}\Pi_{RT}\sigma|_K&=\sum_{\gamma\subset\partial k}\frac{1}{|k|}\int_\gamma n\cdot\sigma ds\\&=\frac{1}{|k|}\int_{\partial k}n_k\cdot\sigma ds\\&=\frac{1}{|k|}\int_K\text{div}\sigma\\&=\Pi_0\text{div}\sigma\end{aligned}$$

where $\Pi_0 : L^2(k) \to \mathbb{P}_0$ is the average value, ie is the L^2 projection. In other words, we have $\text{div}(\Pi_{RT}\sigma) = \Pi_0(\text{div}\sigma)$ and thus we have the following commutative diagram:

$$\begin{array}{ccc} H(\text{div}, k) & \xrightarrow{\text{div}} & L^2(k) \\ \downarrow \pi_0 & & \downarrow \pi_0 \\ \mathbb{RT}_0 & \xrightarrow{\text{div}} & \mathbb{P}_0 \end{array}$$

- There exists a constant $C > 0$ independent of h (the width of elements) such that

$$||\Pi_{RT}\sigma||_K \leq C\{||\sigma||_K + h_k||\nabla\sigma||_K\}.$$

To show this, let $\sigma \in H^1(k)^d$. Then

$$\begin{aligned} \left|\int_\gamma n \cdot \sigma ds\right| &\leq |\gamma| \int_\gamma (n \cdot \sigma)^2 ds \\ &\leq |\gamma| \cdot ||\sigma||^2_{L^2(\gamma)} \end{aligned}$$

which implies that

$$\begin{aligned} \sum_\gamma \left|\int_\gamma n \cdot \sigma ds\right| &\leq \sum_\gamma |\gamma|^{1/2} ||\sigma||_{L^2(\gamma)} \\ &\leq \left(\sum_\gamma |\gamma|\right)^{1/2} \left(\sum_\gamma ||\sigma||^2_{L^2(\gamma)}\right)^{1/2} \\ &= |\partial k|^{1/2} ||\sigma||_{L^2(\partial K)} \end{aligned}$$

where we have used the Cauchy-Schwarz inequality. This gives a version of the trace theorem. Let $\rho(x) = x - x_k$ where x_k is the centroid of element k such that the following hold

  - $\text{div}\rho(x) = d$
  - $n_k \cdot f \geq ch_k$ (shape regularity of triangles - excludes degenerate cases)
  - $|\rho(x)| \leq h_k$ for all $x \in k$.

Then

$$\begin{aligned} ch_k \int_{\partial K} \sigma_1^2 &\leq \int_{\partial k} n_k \cdot \rho\sigma_1^2 \\ &= \int_k \text{div}(\rho\sigma_1^2) \\ &= d\int_k \sigma_1^2 + 2\int_k (\rho \cdot \nabla\sigma_1)\sigma_1 \\ &\leq d||\sigma_1||_k^2 + 2h_k \int_k |\nabla\sigma_1| \cdot \sigma_1 \\ &\leq d||\sigma_1||_k^2 + 2h_k||\sigma_1||_k \cdot ||\nabla\sigma_1||_k \\ &\leq C||\sigma_1||_k(||\sigma_1||_k + h_k||\nabla\sigma_1||_k)^2 \end{aligned}$$

This implies that

$$Ch_k||\sigma_1||^2_{\partial k} \leq C(||\sigma_1||_k + h_k||\nabla\sigma_1||_k)^2$$

and consequently

$$\sum_{\gamma \subset \partial k} \left|\int_\gamma n \cdot \sigma ds\right| \leq C\sqrt{\frac{|\partial k|}{h_k}}\{||\sigma||_k + h_k||\nabla\sigma||_k\}$$

from which it follows that

$$||\Pi_{RT}\sigma|| \leq \sum_{\gamma \subset \partial k} \left| \int_\gamma n \cdot \sigma_k \right| \cdot ||\varphi_\gamma||_k$$

and since

$$||\varphi_\gamma||_k \leq \frac{ch_k|k|^{1/2}}{|k|} = \frac{ch_k}{|k|^{1/2}}$$

it follows that

$$||\Pi_{RT}\sigma|| \leq \frac{Ch_k}{\sqrt{|k|}}\sqrt{\frac{|\partial k|}{h_k}}(||\sigma||_k + ||\nabla\sigma||_k)$$

we also have the estimates

$$|\partial k|^{1/2} \sim h_k^{(d-1)/2}, |k| \sim h_k^{d/2}$$

so

$$||\Pi_{RT}\sigma|| \leq C(||\sigma||_k + h_k||\nabla\sigma||_k).$$

Lecture 9 11/2:

We start with a review of Raviart-Thomas elements. After reviewing the definition of the local basis functions, we go to the globally defined basis functions for $d = 2$. If we have two adjacent elements L and R in $d = 2$, to ensure that normal components are equal (and orienting correctly), we have that

$$\varphi_\gamma = \begin{cases} \frac{1}{2|L|}(x - x_L) & \text{on } L \\ \frac{-1}{2|R|}(x - x_R) & \text{on } R \end{cases}$$

in particular, in this way, basis functions are supported on pairs of elements now. In addition to the 3 properties of RT elements shown in the previous lecture, we prove the following: there exists $C > 0$ independent of h_k such that for all $\sigma \in H^1(k)^d$,

$$||\text{div}(\Pi_{RT}\sigma)||_K \leq C||\text{div}\sigma||_K$$

From the commutative diagram property,

$$||\text{div}(\Pi_{RT}\sigma)||_K = ||\Pi_0(\text{div}\sigma)||_K \leq ||\text{div}\sigma||_K$$

in particular, this shows that RT interpolants are projectors (an idempotent operator on a Hilbert space is a projector if and only if it has a norm of 1). But we had also seen that

$$||\Pi_{RT}\sigma||_{H(\text{div})} \leq C||\sigma||_{H^1(\Omega)^d}.$$

We also give an approximation property of $\mathbb{RT}_0$. In particular, Π_{RT} reproduces some polynomials. Let $p \in \mathbb{P}_0^d$ be arbitrary, then $\Pi_{RT}p = p$. To see this,

$$p = \sum c_\gamma\varphi_\gamma = \sum \varphi_\gamma \int_\gamma n \cdot \tau_\gamma = \Pi_{RT}p.$$

For $\sigma \in H^1(K)^d$, we have

$$\begin{aligned} ||\sigma - \Pi_{RT}\sigma||_K &= ||(\sigma - p) - \Pi_{RT}(\sigma - p)||_K \\ &\leq ||\sigma - p||_K + ||\Pi_{RT}(\sigma - p)||_K \\ &\leq ||\sigma - p||_K + C\{||\sigma - p||_K + h_k||\nabla(\sigma - p)||_K\} \end{aligned}$$

Now choose $p = \frac{1}{|K|}\int_K \sigma dx \in \mathbb{P}_0^d$ and using Poincare, we obtain

$$||\sigma - p||_K \leq Ch_k||\nabla\sigma||_K$$

and hence

$$||\sigma - \Pi_{RT}\sigma||_K \le Ch_k||\nabla\sigma||_K.$$

So these are good approximators in L^2.

Next, consider $\text{div}(\sigma - \Pi_{RT}\sigma) = \text{div}\sigma - \Pi_0\text{div}\sigma$ and so

$$\begin{aligned} ||\text{div}(\sigma - \Pi_{RT}\sigma)|| &= ||\text{div}\sigma - \Pi_0\text{div}\sigma|| \\ &= \inf_{c\in\mathbb{R}} ||\text{div}\sigma - c||_K \\ &\le Ch_K|\text{div}\sigma|_{H^1(K)} \end{aligned}$$

Hence, summing across all $k \in P_h$, we have

$$\begin{aligned} ||\sigma - \Pi_{RT}\sigma|| &\le \left(\sum_{K\in P_h} ||\sigma - \Pi_{RT}\sigma||_K^2\right)^{1/2} \\ &\le C\left(\sum_{k\in P_h} h_k^2||\nabla\sigma||^2_{H^1(K)}\right)^{1/2} \\ &\le Ch||\nabla\sigma||_{H^1(\Omega)} \end{aligned}$$

and similarly,

$$\begin{aligned} ||\text{div}(\sigma - \Pi_{RT}\sigma)||_\Omega &= \left(\sum_{k\in P_h} ||\text{div}(\sigma - \Pi_{RT}\sigma)||_K^2\right)^{1/2} \\ &\le Ch|\text{div}\sigma|_{H^1(\Omega)}. \end{aligned}$$

Combining these, we have

$$||\sigma - \Pi_{RT}\sigma||_{H(\text{div})} \le Ch\{|\sigma|_{H^1(\Omega)} + |\text{div}\sigma|_{H^1(\Omega)}\}$$

Also a key fact is that

$$\left|\int_\gamma n\cdot\sigma\right|$$

is well defined for all $\sigma \in H^1(K)^d$.

We now consider an application to a dual mixed form problem. We are given the continuous problem to find $(\sigma, u) \in H(\text{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\sigma,\tau) + (\text{div}\tau, u) &= 0 \\ (\text{div}\sigma, v) &= -(f, v) \end{aligned}$$

for all $(\tau, v) \in H(\text{div})\times L^2(\Omega)$. The associated $\mathbb{RT}_0\times\mathbb{P}_0$ scheme has the same problem just with $V_h\times M_h = \mathbb{RT}_0\times\mathbb{P}_0$. We start by showing that this scheme satisfies the Brezzi conditions. Ellipticity on the kernel

$$V_h^0 = \{\tau_h \in V_h : (\text{div}\tau_h, v_h) = 0 \quad \forall v_h \in M_h\}$$

But ellipticity follows by choosing $v_h = \text{div}\tau_h \in M_h$. In particular, $a(\tau_h,\tau_h) = ||\tau_h||^2 = ||\tau_h||^2_{H(\text{div})}$ for all $\tau_h \in V_h^0$.

For the discrete inf-sup condition, let $v_h \in M_h$ be piecewise constant and we want to show that there exists $\sigma_h \in V_h$ such that

$$\frac{(\text{div}\sigma_h, v_h)}{||\sigma_h||_{H(\text{div})}} \ge C||v_h||$$

Arguing as in the continuous case, let $\varphi \in H_0^1(\Omega) : -\Delta\varphi = v_h$ in Ω (where we assume Ω is convex, by an extension of domain if necessary). So $\varphi \in H^2(\Omega)$ and let $\sigma = -\operatorname{grad}\varphi \in H^1(\Omega)$ with $||\sigma||_{H^1} \le C||\varphi||_{H^2} \le C||v_h||_{L^2}$. So $\sigma \in H^1(\Omega)$ with $||\sigma||_{H^1(\Omega)} \le C||v_h||_{L^2(\Omega)}$. This means that $\Pi_{RT}\sigma$ is well-defined, so choose this. And moreover, since we are dealing with constant functions, we can remove projection operators so

$$(\text{div}\Pi_{RT}\sigma, v_h) = (\Pi_0\text{div}\sigma, v_h) = (\text{div}\sigma, v_h) = (v_h, v_h) = ||v_h||^2_{L^2}.$$

In addition, $||\Pi_{RT}\sigma||_{H(\text{div})} \leq C||\sigma||_{H^1(\Omega)} \leq C||v_h||_{L^2}$ and hence

$$\frac{(\text{div}\sigma_h, v_h)}{||\sigma_h||_{H(\text{div})}} \geq \frac{||v_h||^2_{L^2}}{C||v_h||_{L^2}} = \frac{1}{C}||v_h||_{L^2}$$

and so inf-sup holds. Finally, by Brezzi there exists a unique solution (σ_h, u_h) and they satisfy the following error estimate:

$$\begin{aligned} ||\sigma - \sigma_h||_{H(\text{div})} + ||u - u_h||_{L^2} &\leq C\{||\sigma - \Pi_{RT}\sigma||_{H(\text{div})} + ||u - \Pi_0 u||_{L^2}\} \\ &\leq Ch\{|\sigma|_{H^1} + |\text{div}\sigma|_{H^1} + ||\nabla||_{L^2}\} \\ &\leq Ch\{||u||_{H^2} + ||f||_{H^1}\} \end{aligned}$$

We then set-up a mixed FEM scheme for Stokes flow given by

$$\begin{cases} -\Delta u + \text{grad}\, p &= f \\ \text{div} u &= 0 \end{cases}$$

in Ω and $u = 0$ on $\partial\Omega$. u is the velocity field, p is the pressure. To make p unique, we require the condition $\int_\Omega p(x)dx = 0$. We choose the variational form

$$\begin{aligned} (\nabla u, \nabla v) - (\text{div} v, p) = (f, v) \\ -(\text{div} u, q) = 0 \end{aligned}$$

for all $(v, q) \in V \times M$ where

$$\begin{aligned} V = \{v \in H^1(\Omega)^d : v = 0 \text{ on } \partial\Omega\} = H^1_0(\Omega)^d \\ M = \{q \in L^2(\Omega) : \int_\Omega q = 0\} \end{aligned}$$

We show that this is well-posed. Assume $f \in L^2(\Omega)^d$ and we know that V, M are Hilbert spaces. Let

$$V_0 = \{v \in V : (\text{div} v, q) = 0 \quad \forall q \in L^2(\Omega)\} = \{v \in V : \text{div} v = 0\}$$

But $a(\cdot, \cdot)$ is elliptic on $H^1_0(\Omega)^2$ and so in particular on V_0. The inf-sup condition can also be shown. There exists $v \in V$ such that if Ω is bounded and connected, then $\text{div} v = q$ in Ω and $||v||_V \leq C||q||_M$. Which implies inf-sup

$$\frac{(\text{div} v, q)}{||v||_V} \geq \frac{||q||^2_M}{C||q||_M} \geq \frac{1}{C}||q||_M$$

which is the sense of div having a continuous right inverse.

Lecture 10 11/9

We go back to analyzing Stokes flow given by

$$\begin{cases} -\Delta u + \nabla p &= f \\ \text{div} u = 0 \end{cases}$$

subject to $u = 0$ on $\partial\Omega$ and we impose the compatibility condition $\int_\Omega p dx = 0$. The mixed variational form of this is solving for $(u, p) \in V \times M$ for

$$\begin{cases} (\nabla u, \nabla v) - (\text{div} v, p) &= (f, v) \\ -(\text{div} u, q) = 0 \end{cases}$$

for all $(v, q) \in V \times M$ where $V = H^1_0(\Omega)^2, M = \{q \in L^2(\Omega) : \int_\Omega q = 0\} = L^2 \setminus \mathbb{R}$. This is well-posed if the inf-sup condition holds. etc etc... Suppose we choose a triangular partition with degrees of freedom at vertices by doing H^1 interpolation with

$$V_h = \{v \in V : v|_k \in P_1 \times P_1 \quad \forall k \in P_h\}$$

and for L^2 interpolation, choose

$$M_h = \{q \in M : q|_k \in P_0 \quad \forall k \in P_h\}$$

where we recall $P_1 = \text{span}\{1, x, y\}$. This is a classical $P_1^2 - P_0$ elements. We start with a dimension count. $\dim V_h = 2v_1$ where v_1 is the number of interior vertices on the mesh while $\dim M_h = t-1$ where t is the number of elements in the partition. Adding an interior vertex increases v_I by 1 while increasing t by 2. Adding a boundary vertex increases v_B by 1 while increasing t by 1. This suggests that $t - 2v_I - v_B$ is constant. In particular, $t - v_I - v_B = -2$ (maybe this has something to with Euler characteristics!). But $\dim M_h = t - 1 = 2v_I + v_B - 3 = \dim v_H + (v_b - 3)$ where the last term is positive if $t > 1$ (trivial). This implies that

$$\dim M_h > \dim V_h > \dim \text{div} V_h.$$

The method is not stable.

The previous suggests that considering degrees of freedom at vertices is not the best idea. So let's consider degrees of freedom at edges. In this case, $\dim M_h = t-1$ while $\dim V_h = e_I = e - v_B = v + t - 2 - v_B = v_I + t - 2 \geq t - 1$ for $v_I \geq 1$. Here e_I is the number of interior edges. This is a potentially stable method, but is probably a bad idea since $V_h \subset V = H_0^1$ and RT_0 is for $H(\text{div})$ which means there are only constrains on the normal components between edges.However, making RT_0 continuous means that $V_h = \{0\}$. If we try again by choosing $V_h = P_1^{CR} \times P_1^{CR}$, $M_h = P_0$ (this forms the Crouszeix-Raviart element) tat requires continuity at midpoints of edges rather than at vertices, we get a non-confirming element, as $\dim V_h^{CR} = 2e_I, \dim M_h = t-1$ and $\dim V_h^{CR} > \dim M_h$ for $V_i \geq 1$. We could relax the conforming condition and make a compromise by not requiring that $V_h \subset V$.

We go through some of the properties of Crouzeix-Raviart elements. By choosing the degrees of freedom at midpoints of a triangular elements, the usual barycentric argument gives the basis functions

$$\{\varphi_i\} = \{1 - 2\lambda_1, 1 - 2\lambda_2, 1 - 2\lambda_3\}$$

which are in P_1. We also have that (by defining γ_i as the edges of the triangle),

$$\frac{1}{|\gamma'|} \int_{\gamma'} \varphi_\gamma ds = \delta_{\gamma\gamma'}.$$

We also have $||\nabla \varphi_\gamma|| \sim C$ independent of h. The CR interpolant corresponds to the degrees of freedom

$$v \mapsto \frac{1}{|\gamma|} \int_\gamma v ds \quad \forall v \subset \partial K$$

and so

$$\Pi_h^{CR} v = \sum_{\gamma \subset \partial K} \varphi_\gamma \left(\frac{1}{|\gamma|} \int_\gamma v ds \right)$$

We get continuity across elements at the midpoints of edges and have the following relation

$$\frac{1}{|\gamma|} \int_\gamma \Pi_h^{CR} v ds = \frac{1}{|\gamma|} \int_\gamma v ds$$

(something something Marini equivalence between CR and RT elements) We give a few more properties. First, stability. Let $v \in H^1(K)$, then

$$\left|\Pi_h^{CR} v\right|_{H^1} \leq \sum_{\gamma \subset \partial K} |\nabla \varphi_\gamma|_K \left| \frac{1}{|\gamma|} \int_\gamma v ds \right| \leq C \sum_{\gamma \subset \partial K} \left| \frac{1}{|\gamma|} \int_\gamma v ds \right|$$

this operator is bounded with

$$\left|\Pi_h^{CR} v\right|_{H^1} \leq C\{h_k^{-1} ||v||_K + |v|_{H^1}\}$$

It also satisfies the commuting property with divergence, ie $\text{div}\Pi_h^{CR} = \Pi_0 \text{div}$.

We want an approximation property of P_1. Let $v \in H^1$ and choose $v_h \in V_h^{\text{conforming}}$ (eg take the L^2 projection of v onto V_h^{cone}), then

$$\sum_{k \in P_h} \{h_k^{-1} ||v - v_h|| + |v - v_h|_{H^1}\} \leq C|v|_{H^1}.$$

Let $v^{CR} = \Pi^{CR} v$ then

$$||v^{CR}||_{H^1} \leq ||\Pi^{CR} v||_{H^1} \leq C(h^{-1}||v||_{L^2} + ||v||_{H^1}) \leq Ch^{-1}||v||_{H^1} \leq C\beta h||q_h||_{L^2}.$$

To deal with the instability caused by the h^{-1} factor, we construct a **Fortin operator** $\Pi_h^F : V \to V_h^{CR}$ by

$$\Pi_h^F v = \Pi_h^{CR}(v - v_h).$$

Stability goes as follows

$$\begin{aligned} |\Pi_h^F v|_{H^1} &\leq |v_h|_{H^1} + |\Pi^{CR}(v - v_h)|_{H^1} \\ &\leq |v_h|_{H^1} + C \sum_{k \in P_h} \{h_k^{-1}||v - v_h||_k + |v - v_h|_{H^1}\} \\ &\leq |v_h|_{H^1} + C|v|_{H^1} \\ &\leq |v - v_h|_{H^1} + (C+1)|v|_{H^1} \\ &\leq \tilde{C}|v|_{H^1} \end{aligned}$$

This operator also satisfies the commuting diagram property. Let $q_h \in M_h$ then

$$\int_\Omega \text{div}(\Pi^F v) q_h = \int_\Omega (\text{div} v_h) q_h + \int_\Omega \text{div}\Pi_h^{CR}(v - v_h) q_h = \int_\Omega (\text{div} v_h) q_h.$$

Now we show inf-sup. Let $q_h \in M_h$ be given. By Ladyzhenskaya, there exists $v \in H^1(\Omega)^2$ such that

$$\int_\Omega (\text{div} v) q_h = ||q_h||_M^2$$

and is a continuous right inverse with $||v||_V \leq \frac{1}{\beta}||q_h||_M$. Choose $v_h^{CR} \in V_h^{CR}$ such that $v_h^{CR} = \Pi_k^F v$. Then

$$\int_\Omega (\text{div} v_h^{CR}) q_h = \int_\Omega (\text{div}\Pi_h^F v) q_h = \int_\Omega (\text{div} v) q_h = ||q_h||_M^2$$

and

$$||v_h^{CR}||_V = ||\Pi_h^F v||_{H^1(\Omega)^2} \leq C||v||_{H^1(\Omega)^2} \leq \frac{C}{\beta}||q_h||_M$$

hence

$$\frac{(\text{div} v_h^{CR}, q_h)}{||v_h^{CR}||_V} \geq \frac{\beta}{C}||q_h||_M.$$

We extract the general principle of Fortin's lemma as follows.

Suppose $b : V \times M \to \mathbb{R}$ satisfies the inf-sup condition. Let $\Pi^F : V \to V_h$ be such that

- i) $||\Pi^F v||_V \leq C||v||_V \quad \forall v \in V$
- ii) $b(\Pi^F v, q_h) = b(v, q_h) \quad \forall q_h \in M_h$

Then the pair $V_h \times M_h$ is inf-sup stable with constant $C\beta$.

As proof, let $q_h \in M_h \subset M$. Then by the continuous inf-sup condition, there exists $v \in V$ such taht

$$b(v, q_h) = ||q_h||_M^2, \quad ||v||_V \leq \beta||q_h||_M$$

Choose $v_h = \Pi_h^F v \subset v_h$. Then

$$b(v_h, q_h) = b(\Pi_h^F v, q_h) = b(v, q_h) = ||q_h||_M^2$$

and

$$||u_h||_V = ||\Pi^F v||_V \leq C||v||_V \leq C\beta||q_h||_M.$$

This technique is used to show stability for schemes for Stokes problems and leads to things like the Taylor-Hood elements...

Lecture 11 11/16

We now consider the finite element discretization of $H(\text{curl})$. Let $\Omega \subset \mathbb{R}^3$ be a sufficiently smooth domain to justify the following operations. Then define

$$H(\text{curl};\Omega) = \{v : \Omega \to \mathbb{R}^3; v, \nabla \times v \in L_2(\Omega)^3\}$$

with the norm given by

$$||v||^2_{H(\text{curl})} = ||v||^2_{L^2} + ||\nabla \times v||^2_{L^2}.$$

Similar to $H(\text{div})$ we consider inter-element continuity in $H(\text{curl})$. Consider a plane that cuts the 3D domain and a ball that is split into $\Omega_1 \cup \Omega_2$ where $F = \partial\Omega_1 \cap \partial\Omega_2$ is the boundary. Let $\psi \in H^1_0(B)$. Then

$$\begin{aligned}
0 &= \int_{\partial B} \psi n \times v ds \\
&= \int_{\Omega} \text{curl}(\psi v) dx \\
&= \int_{\Omega_1} \text{curl}(\psi v) + \int_{\Omega_2} \text{curl}(\psi v) \\
&= \int_{\partial\Omega_1} \psi n_1 \times v|_{\Omega_1} + \int_{\partial\Omega_2} \psi n_2 \times v|_{\Omega_2} \\
&= \int_{F} \psi n_1 \times v|_{\Omega_1} + \int_{F} \psi n_2 \times v|_{\Omega_2} \\
&= \int_{F} \psi [n \times v] dx \quad \forall \psi \in H^1_0(\Omega).
\end{aligned}$$

This motivates the new notion of the jump $[n \times v]$ and this gives the condition under which continuity is preserved: when tangential components of the vector fields are continuous. Hence if ψ is sufficiently smooth, require $[n \times v]|_F = 0$ for all F, meaning tangential components are continuous in the sense that limits from the left and from the right are equal (note that there are no point-wise evaluations).

We now want to show that $n \times v$ is well-defined as the above were formal manipulations when $v \in H(\text{curl};\Omega)$. By well-defined here, we mean in the distributional sense. We show the following lemma:

Let $v \in C^\infty(\Omega)^3$ (this suggests that we will be making a density argument). Then

$$\begin{aligned}
\int_{\partial\Omega} \psi n \times v ds = \int_\Omega \text{curl}(\psi v) &= \int_\Omega \nabla\psi \times v + \int_\Omega \psi \,\text{curl}\, v \\
&\le ||\nabla\psi|| \cdot ||v|| + ||\psi|| \cdot ||\,\text{curl}\, v|| \\
&\le ||\psi||_{H^1} \cdot ||v||_{H(\text{curl})}
\end{aligned}$$

by an application of Cauchy-Schwarz. Hence hte map is continuous for $v \in H(\text{curl})$ and linearity is obvious.

We now consider the construction of finite element subspaces of $H(\text{curl};\Omega)$. Let P_h be a partitioning of Ω into tetrahedra under the usual conditions (where things are non-degenerate, they share either a common face, or edge, or vertex). Consider the following:

- $H(\text{div})$ has continuity on faces of normal components
- $H(\text{curl})$ has continuity on faces and edges for tangential components
- $H^1(\Omega)$ has continuity on faces, edges and vertices for all values

This short summary follows from previous discussions and serves as a motivating set of results for the following discussion. We observe that as we move up the list from the bottom, we take away the lowest dimensional object at each step (this can be thought of in terms of differential forms somehow). So, we need to define basis functions with continuous tangential components on faces and edges. Choose degrees of freedom associated with tangential components on edges. We expect 6 degrees of freedom since there are 6 edges and expect that $\dim P = 6, \dim\Sigma = 6$ for unisolvent Σ, P.

We first consider a simpler case where $d = 2$ and

$$H(\text{curl}) = \{v : \Omega \to \mathbb{R}^2, v \in L^2(\Omega)^2\}.$$

In particular, in 2D, $v \in H(\text{curl}) \iff v^\perp \in H(\text{div})$ and $\text{curl} = \nabla \cdot (\cdot)^\perp$, $\text{div} = \nabla \cdot (\cdot)$. From before, we know how to discretize the space $H(\text{div})$ in 2D by $\mathbb{RT}_0$. But this gives that $v^\perp$ are continuous normals while we want v continuous tangential components. This motivates rotating 2D Raviart-Thomas elements to construct new basis functions. In particular,

$$\varphi_\gamma^{\text{curl}} = (\varphi_\gamma^{RT})^\perp = \frac{1}{2|K|}(x - x_\gamma)^\perp.$$

However, this line of reasoning will not generalize to 3D since those basis functions are vertex-based and we are interested in edge-based formulations of basis functions. We claim that through barycentric coordinates λ_ℓ, λ_r we can express the basis functions for an edge γ going left-right,

$$\varphi_\gamma^{RT} = \lambda_\ell (\nabla \lambda_r)^\perp - \lambda_r (\nabla \lambda_\ell)^\perp.$$

This identity transforms from vertex-based to edge-based basis functions in terms of endpoints of edges. We take the Whitney 1-forms, of which there are 6 in three dimensions:

$$\varphi_\gamma = \lambda_\ell \nabla \lambda_r - \lambda_r \nabla_\ell$$

As a sidenote, the nice thing about barycentric coordinates is that they see all dimensions of subsimplices in the sense that the barycentric property is maintained when going down dimensions of the subsimplices (ie true on tetrahedron, true on a particular face, true on a particular edge, etc... but this is a general property of barycentric coordinates for simplicial complexes).

Let x_0, x_1, x_2, x_3 define the vertices of a tetrahedron. We check tangential continuity of the form

$$\varphi_{01} = \lambda_0 \nabla \lambda_1 - \lambda_1 \nabla \lambda_0$$

We consider the tangential component of φ_{01} on the face $F_{023} = \Delta(x_0, x_2, x_3)$. Then $\nabla \lambda_1|_{F_{023}}$ is proportional to the normal of the face and in particular we have

$$\varphi_{01}|_{F_{023}} = \lambda_0 \nabla \lambda_1|_{F_{023}}.$$

Hence φ_{01} has vanishing tangential components on F_{023}. A symmetrical observation gives the same vanishing tangential component property on F_{123}. Additionally, we have a vanishing tangent on all edges except for the edge $[x_0, x_1]$. Hence we obtain

$$\int_{\gamma'} t_{\gamma'} \cdot \varphi_\gamma ds = 0 \quad \forall \gamma \neq \gamma'.$$

Now consider that for the corresponding tangential component, we have

$$\begin{aligned} t_\gamma \cdot \varphi_\gamma &= -t_\gamma(\lambda_r \nabla \lambda_\ell = \lambda_\ell \nabla \lambda_r) \\ &= \lambda_\ell \cdot \frac{1}{|\gamma|} + \lambda_r \cdot \frac{1}{|\gamma|} \\ &= \frac{1}{|\gamma|}(\lambda_\ell + \lambda_r). \end{aligned}$$

As such

$$\int_\gamma t_\gamma \cdot \varphi_\gamma ds = \int_\gamma \frac{1}{|\gamma|} ds = 1$$

so we obtain the unisolvence property again and inparticular we have linear independence of the proposed basis functions and in particular $\dim \Sigma = 6$ and in particular

$$\int_{\gamma'} t_{\gamma'} \cdot \varphi_\gamma ds = \delta_{\gamma\gamma'}.$$

So we have shown that

- $\{\varphi_\gamma\}$ are linearly independent
- $\dim P = 6$
- $\{\varphi_\gamma\}$ are unisolvent with respect to

$$\Sigma = \left\{ v \mapsto \int_\gamma t_\gamma \cdot v ds, \gamma \in \mathcal{E} \right\}$$

Taken in combination, the above implies that the triple (Tet, Σ, P) forms a finite element. These are actually called Nedelec elements.

As an interlude, consider a tetrahedron in the simplex configuration. Then $\lambda_1 = x, \lambda_2 = y$ and

$$\varphi_{12} = \lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1 = x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - y \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

Then $x \cdot \varphi_\gamma \in \mathbb{P}_1$ for all γ if and only if $\varphi_\gamma = \mathbb{P}_0^3 +$ "Rotation of x$''$. And in particular, this means that

$$P = \mathbb{P}_0^3 + x \times \mathbb{P}_0^3 = \mathbb{ND}_0.$$

This yields the following formal definition of a Nedelec element. Let $K = \text{Tet}$, P be as above and

$$\Sigma = \left\{ v \mapsto \int_\gamma t_\gamma \cdot v ds, \gamma \in \mathcal{E} \right\}$$

We now define a Nedelec interpolant by

$$\Pi_N v = \sum_{\gamma \in \mathcal{E}_K} \varphi_\gamma \int_\gamma t_\gamma \cdot v ds$$

on K for sufficiently smooth v. Note that

$$\int_\gamma t_\gamma \cdot \Pi_N v ds = \int_\gamma t_\gamma \cdot v ds \quad \forall \gamma \in \mathcal{E}.$$

This gives us part of a commutative square. But we are missing the bottom-left term. To investigate this, let $w \in H^1$ be sufficiently smooth so that

$$\begin{aligned} \Pi_N \text{grad} w &= \sum_{\gamma \in \mathcal{E}} \varphi_\gamma \int_\gamma t_\gamma \cdot \text{grad} w ds \\ &\leq \sum_{\gamma \in \mathcal{E}} \varphi_\gamma (w(x_r) - w(x_\ell)) \\ &= \sum_{\ell < r} (\lambda_\ell \nabla \lambda_r - \lambda_r \nabla \lambda_\ell)(w_r - w_\ell). \end{aligned}$$

The coefficient for w_0 comes from considering the edges $(0,1), (0,2), (0,3)$. In particular after some algebra we get that the coefficient of w_0 is $\nabla \lambda_0$. Hence

$$\Pi_N \text{grad} w = \nabla (w_0 \lambda_0 + w_1 \lambda_1 + w_2 \lambda_2 + w_3 \lambda_3) = \text{grad}(\Pi w)$$

where Π is the interpolation operator. As such we get the following de Rham complex that is known as the Whitney complex in the literature:

$$\begin{array}{ccccccc} H^1(K) & \xrightarrow{\nabla} & H(\text{curl}) & \xrightarrow{\nabla \times} & H(\text{div}) & \xrightarrow{\nabla \cdot} & L^2 \\ \downarrow \Pi & & \downarrow \Pi_N & & \downarrow \Pi_{RT} & & \downarrow \Pi_0 \\ \mathbb{P}_1(K) & \xrightarrow[\nabla]{} & \mathbb{ND}_0 & \xrightarrow[\nabla \times]{} & \mathbb{RT}_0 & \xrightarrow[\nabla \cdot]{} & \mathbb{P}_0(K) \end{array}$$

In particular, we have the property that

$$\nabla \times (\Pi_N v) = \sum_{\gamma \in \mathcal{E}} \alpha_\gamma \nabla \times \varphi_\gamma; \quad \alpha_\gamma = \int_\gamma t_\gamma \cdot \varphi_\gamma ds$$

and moreover

$$\begin{aligned} \nabla \times \varphi_\gamma &= \nabla \times (\lambda_m \nabla \lambda_n - \lambda_n \nabla \lambda_m) \\ &= 2\nabla \lambda_m \times \nabla \lambda_n \\ &= \frac{1}{3|K|}(x_\ell - x_0) \quad \gamma = (0, \ell) \end{aligned}$$

Hence

$$\begin{aligned} n_0 \cdot \nabla \times \varphi_\gamma &= \frac{1}{3|K|} n_0 \cdot (x_\ell - x_0) \\ &= \frac{1}{3|K|} \text{dist}(x_0, F_0) \\ &= \frac{1}{|F_0|}. \end{aligned}$$

It follows that

$$n_0 \cdot \nabla \times (\Pi_N v)|_{F_0} = \frac{1}{|F_0|} \int_{\partial F_0} t \cdot v ds = \frac{1}{|F_0|} \int_{F_0} n \times \nabla \times v ds$$

and so

$$\int_{F_0} n_0 \cdot \nabla \times (\Pi_N v)|_{F_0} = \int_{F_0} n \cdot (\nabla \times v)$$

So $\nabla \times (\Pi_N v) \in \mathbb{P}_0^3$ has the same normal components as $\nabla \times v$.

Lecture 12 11/30

This is the final lecture of the course that introduces new material, whereas the true last lecture of the semester is by Charlie Parker, a recently-graduated PhD from Brown.

Recall that we have constructed an exact sequence for $\Omega \subset \mathbb{R}^3$:

$$\begin{array}{ccccccc} H^1(K) & \xrightarrow{\nabla} & H(\text{curl}) & \xrightarrow{\nabla\times} & H(\text{div}) & \xrightarrow{\nabla\cdot} & L^2 \\ \downarrow \Pi & & \downarrow \Pi_N & & \downarrow \Pi_{RT} & & \downarrow \Pi_0 \\ \mathbb{P}_1(K) & \xrightarrow[\nabla]{} & \mathbb{ND}_0 & \xrightarrow[\nabla\times]{} & \mathbb{RT}_0 & \xrightarrow[\nabla\cdot]{} & \mathbb{P}_0(K) \end{array}$$

Here, the bases are given by $\{1, x, y, z\}$ corresponding to interpolation over vertices in a mesh, $\{a + b \times x\}$ corresponding to interpolation of tangential components of edges in a mesh, $\{a + bx\}$ corresponding to interpolation of normal components on faces, and finally $\{1\}$ corresponding to averages over the interior of elements in the mesh. We have also shown the following inequalities for the above interpolation operations:

$$||u - \Pi u|| \leq Ch||u||_1, \quad ||u - \Pi_N u||_{H(\text{curl})} \leq Ch\{||u||_1 + ||\operatorname{curl} u||_1\}, \quad ||u - \Pi_{RT} u||_{H(\text{div})} \leq Ch\{||u||_1 + ||\text{div} u||_1\}$$

respecitvely.

We now apply this to a magnetostatics problem. We seek $(u, p) \in V \times M$ such that

$$\begin{aligned} (\nabla \times u, \nabla \times v) + (v, \nabla p) &= (J, v) \\ (u, \nabla q) &= 0 \end{aligned}$$

for all $(v, q) \in V \times M$ where $J \in L^2$ is the current density, though it could be given in the dual space $H^{-1}(\text{curl})$. Here, we have the essential condition given in the definition of the two spaces:

$$V = \{v \in H(\text{curl}; \Omega) : n \times v = 0 \quad \text{on } \partial\Omega\}$$
$$M = H_0^1(\Omega)$$

Correspondingly, we choose the discrete spaces

$$V_h = \{v \in V : v|_K \in \mathbb{ND}_0(K) \quad \forall K \in P_h\}$$
$$M_h = \{q \in H_0^1 : q|_K \in \mathbb{P}_1(K) \quad \forall K \in P_h\}$$

We show stability of the associated discrete scheme. Let $q_h \in M_h$ be given and choose $v_h = \text{grad}\, q_h \in V_h$. Then

$$\frac{(v_h, \text{grad}\, q_h)}{||v_h||_{H(\text{curl})}} = \frac{||\,\text{grad}\, q_h||^2}{||\,\text{grad}\, q_h||_{H(\text{curl})}} = \frac{||\,\text{grad}\, q_h||^2}{||\,\text{grad}\, q_h||} = ||\,\text{grad}\, q_h|| = ||q_h||_M$$

The above immediately gives inf-sup stability and the discrete inf-sup constant is the same as the constant for the continuous inf-sup condition. To show ellipticity on the kernel, let $v_h \in V_h^0$ which means $v_h \in V_h$ and $(v_h, \text{grad}\, q_h) = 0$ for all $q_h \in M_h$. We require the estimate

$$||v_h|| \le C_M ||\nabla \times v_h||$$

but this is the discrete Friedrichs inequality. This inequality is sometimes also proven via Bodovski operators. Note that $\nabla \times v_h = 0 \iff v_h = \text{grad}\, q_h \iff v_h = 0$. This gives that

$$a(v_h, v_h) = ||\nabla \times v_h||^2 \ge \frac{1}{1 + C_M}(||\nabla \times v_h||^2 + ||v_h||^2) = \frac{1}{1 + C_M}||v_h||^2_{H(\text{curl})}.$$

Thus, Brezzis' theorem applies and we have the inequality

$$||u - u_h||_v + ||p - p_h||_M \le C\{||u - \Pi_N u||_V\} \le Ch\{||u||_{H^1} + ||\nabla \times u||_{H^1}\}.$$

The second application is to stable mixed methods in incompressible flows. There's a discussion to be had about dimension counting, but we try something called the Taylor-Hood element which is a $\mathbb{P}_2 \times \mathbb{P}_2 - \mathbb{P}_1^{CTS}$ element, where $V_h = \mathbb{P}_2 \times \mathbb{P}_2$ and $M_h = \mathbb{P}_1^{CTS}$ which has $\dim V_h = 10, \dim M_h = 4$. This element was shown to be stable in 2013. The main issue is of showing inf-sup stability. In particular, that for all $p_h \in M_h$, there exists $v_h \in V_h$ such that

$$\frac{(\text{div} v_h, p_h)}{||v_h||_1} \ge \beta ||p_h||_M$$

where $\beta > 0$ is independent of h. We try to construct a Fortin operator (which had been previously used for CR). Let $\Pi^F = \Pi^C + \Pi^H(I - \Pi^c)$ where $\Pi^H : V \to V_h$ is the Ladyzhenskaya velocity that satisfies

- (1) $||\Pi^H v||_{H^1} \le C||v||_{H^1}$ for all $v \in H_0^1 \times H_0^1$
- (2) has the commuting property $(\text{div}\Pi^H, p_h) = (\text{div} v, p_h)$ for all $p_h \in M_h$. This is equivalent, by integration by parts, to $(\Pi^H v, \text{grad}\, p_h) = (v, \text{grad}\, p_h)$ for all $p_h \in M_h$. But this requires continuous pressrues for this

To get this, we use edge degrees of freedom in V_h to construct the Fortin operator. In particular let

$$\Pi^H v = \sum c_\gamma \beta_\gamma t_\gamma$$

where c_γ are constants to be determined, $\beta_\gamma = \lambda_\ell \lambda_r$ are edge bubbles, and t_γ are tangents on edges. Choose $\{c_\gamma : \mathcal{E}_I\}$ are on interior edges and

$$(\Pi^H v, \text{grad}\, p_h) = (v, \text{grad}\, p_h) \quad \forall p_h \in M_h$$

which gives a rectangular linear system in c_γ. Consider $\text{grad}\, M_h \sim \text{grad}\, p_h$. Let $w_h \in \text{grad}\, M_h$ be such that

- $w_h|_K \in \mathbb{P}_0 \times \mathbb{P}_0$

- $\nabla \times w_h|_K = 0$
- $[|t \cdot w_h|]_\gamma = 0$

Which motivates the choice of Nedelec elmeents, which is an interesting choice because this means we should be thinking of $\operatorname{grad} q_h$ not as the gradient of piecewise linear functions, but simply as an element of a subspace of $H(\text{curl})$. In particular $\operatorname{grad} M_h \subset W_h$ where W_h are Nedelec elements and is given by $W_h = \text{span}\{\varphi_\gamma : \gamma \in \mathcal{E}\}$ where φ_γ is a Nedelec basis function. It suffices to show that $\operatorname{grad} \lambda_n \in W_h$ for all $n \in \mathbb{N}$. Consider a single triangle and without loss of generality, let $\lambda_n = x_3$. Thus $\varphi_1 - \varphi_2 = \cdots = \operatorname{grad} \lambda_3$ on K and similarly for other elements, so $\operatorname{grad} \lambda_3 \in W_h$. But this also means that $\dim W_h$ is the number of element edges when we wanted this dimension to equal the number of internal element edges on the mesh.

This means we need to reduce the space W_h such that $\dim W_h = |\mathcal{E}_I|$. To do this, define $\tilde{W}_h = \text{span}\{\tilde{\varphi}_2, \tilde{\varphi}_3\}$ where $\tilde{\varphi}_2 = \varphi_2 - \varphi_1$ and $\tilde{\varphi}_3 = \varphi_3 - \varphi_1$. Observe that $\nabla \lambda_1 = \tilde{\varphi}_1 - \tilde{\varphi}_3$. We have that $\tilde{W}_h \subset W_h$ and is reduced such that $\operatorname{grad} M_h \subset \tilde{W}_h$ with $\dim \tilde{W}_h = \mathcal{E}_I$, but this second condition holds provided that no element has more than one eedge on $\partial\Omega$.

We now go back to the construction of the Fortin operator $\Pi^H v = \sum_{\gamma \in \mathcal{E}_I} c_\gamma t_\gamma \beta_\gamma$ such that

$$(\Pi^H v, w_h) = (v, w_h) \quad \forall w_h \in \tilde{W}_h$$

which now results in a square system of equations. We demonstrate solvability using BBN theory. ie it remains to show that the inf-sup condition holds for the discrete problem Choosing $w_h = \sum_{\gamma \in \mathcal{E}_I} c_\gamma \tilde{\varphi}_\gamma$ we get modified Nedelc elements and in particular it can be shown that

$$(\Pi^H, v, w_h) \geq \frac{1}{30} \sum_{k \in P_h} |K| \sum c_\gamma^2.$$

This gives that the associated matrix is invertible and that there exists a $C > 0$ such that

$$||\Pi^H v||^2 \leq C \sum_{k \in P} |k|^2 \sum c_\gamma^2$$
$$||w_h||^2 \leq C \sum c_\gamma^2$$

Then supposing that the mesh is quasi-uniform, we have $|K| \sim h^2$ and in particular that

$$||\Pi^H v||^2 \leq C h^2 \sum c_\gamma^2$$
$$||w_h||^2 \leq C \sum c_\gamma^2$$

This is in essence a 2013 theorem by Winther, Mardal, and Schobert in Numerische Mathematik: there exists an operator Π^H such that

- $||\Pi^H v||_{H^1} \leq C||v||_{H^1}$
- $(\text{div}\Pi^H v, q_h) = (\text{div} v, q_h)$ for all $q_h \in M_h$

The corollary to this is that the Taylor-Hood element is inf-sup stable if and only if no element has more than one edge on $\partial\Omega$.